Text Is Fly Within The Book Only

CARDINAL ALGEBRAS

To the memory of my friends and students murdered

in Poland during the Second World War

CARDINAL ALGEBRAS

BY

ALFRED TARSKI

• • •

WITH AN APPENDIX

CARDINAL PRODUCTS OF ISOMORPHISM TYPES

BY

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PREFACE

This work has its origin in certain studies in general set theory and, more specifically, in the arithmetic of cardinal numbers.[1]

Among the results which have been obtained so far in the arithmetic of cardinals, two main types can be distinguished. On the one hand, we have a series of very strong and general theorems which exhaust large portions of the arithmetic of cardinals, e.g., the theory of cardinal addition; these theorems have been established by applying the so-called axiom of choice in its most general form and, in particular, the well-ordering principle. As examples we may mention the theorems stating that the cardinals are well ordered and that the sum of any two infinite cardinals equals the larger of them. On the other hand, results are known which are of a more restricted (though by no means more trivial) character, but which have been obtained in a more constructive way, mostly without the help of the axiom of choice; the method involved consists, roughly speaking, in considering directly transformations which establish one-to-one correspondences between given sets and in constructing in terms of these sets and transformations new sets and new transformations. The best-known example is the Cantor-Bernstein equivalence theorem by which any two cardinals are equal if each of them is at most equal to the other.

The results of the second type are interesting not only from the point of view of foundations. By analyzing their proofs we usually arrive at more general formulations which belong to the general

[1] The present work was conceived about twenty years ago, but circumstances beyond the control of the author have rendered its earlier realization impossible. The publications of the author related to the central ideas of this work are listed in the bibliography; in connection with the introduction compare, in particular, Tarski [2] and Tarski [8]. The understanding of this book does not require an extensive knowledge of set theory, but a certain orientation in fundamental notions and methods of this discipline is desirable and will prove helpful in reading the last three sections of the work. For general information in this domain the reader may consult the first few chapters in Hausdorff [1], Schoenflies [1], or Sierpiński [1]. Schoenflies' work contains many valuable historical references, while Sierpiński's book will throw light on the discussions involving the axiom of choice.

theory of one-to-one transformations, and which have found some interesting applications outside the domain of abstract set theory—for instance, in the theory of measure.

All the results of the second type which concern cardinal addition prove to be derivable in a purely arithmetical way from a small number of basic theorems. The derivations are not simple but, in general, are not more involved than direct proofs carried through by the method indicated above. Also some new and interesting results can be obtained in this way. The proof of the basic theorems themselves presents no difficulties. The basic theorems have the character of formal laws which apply not only to the arithmetic of cardinals but also to various other mathematical systems. All this suggests the idea of introducing and studying a new kind of abstract algebras for which these basic theorems would serve as defining postulates. A realization of this idea is the main purpose of the present work.

The algebras in question will be referred to as CARDINAL ALGEBRAS. Each of them is constituted by a set of arbitrary elements and by two operations, that of binary addition and that of addition of infinite sequences. (Each of these operations can be defined in terms of the other.) The algebras are assumed to satisfy a number of familiar laws of an elementary nature—closure postulates, commutative and associative laws, and the postulate of the zero element; and, in addition, two existential postulates which are characteristic of cardinal algebras—the refinement postulate and the remainder, or infinite chain, postulate.

The variety of arithmetical laws which follow from these postulates is very large; and so is the variety of mathematical systems in which these postulates are satisfied. As elementary examples of cardinal algebras we list non-negative integers and non-negative real numbers (under ordinary addition) with ∞ included in both cases; non-negative real functions over an arbitrary domain; countably complete fields of sets (under set-theoretical addition); and—more generally—countably complete Boolean algebras. As examples of a less elementary character we mention cardinal numbers and relation numbers (under cardinal addition); isomorphism types of countably complete Boolean algebras and of certain more general classes of lattices (under direct multiplication); and generalized homeomorphism types of Borelian sets in an arbitrary metric space.

Several general methods are available which permit the construction of new cardinal algebras from given ones; some of them are known from modern algebra—e.g., direct multiplication and homomorphic transformation—while others are specific for the algebraic systems under discussion.[2] A combination of these methods leads, for instance, from fields of sets to cardinal numbers, and from Boolean algebras to isomorphism types of these algebras; and hence difficult theorems on sums of cardinals and on direct products of isomorphism types appear as consequences of the fact that these and similar theorems trivially apply to set-theoretical sums of sets and to least upper bounds of elements in a Boolean algebra.

This work is divided into three parts. The first contains the definition of cardinal algebras and the development of their arithmetic. The second part is concerned with the discussion of general methods of constructing cardinal algebras; moreover, the results are extended to a wider class of algebraic systems (important from the point of view of applications), the so-called generalized cardinal algebras. In the third part we study the connections between cardinal algebras and related algebraic systems, namely, commutative semigroups and lattices; and we discuss various special examples of cardinal algebras. Certain applications of the theory of cardinal algebras to general algebra—in fact, to the theory of direct products of isomorphism types—are discussed in the appendix.

It will be seen from the above description that this work lies within the domain of abstract algebra. Some features of it, however, will probably seem strange to most algebraists—and certainly do not conform to an orthodox algebraic point of view. We have here in mind especially the part played by infinite addition and the application of frankly infinite constructions in arithmetical developments; the discussion of algebras which are not assumed to satisfy closure postulates (the generalized cardinal algebras mentioned above); and—among minor details of a rather terminological nature—the extensive use of the notion of a function.

Many arithmetical results of the first part of the work have been obtained by extending certain known theorems of the arithmetic of cardinals which originate with various authors. Some of these

[2] For general information in the domain of modern algebra consult Birkhoff [3], Birkhoff-MacLane [1], and van der Waerden [1].

theorems, however, can be found only in preliminary reports, and their proof appears here for the first time.[3] In some other cases the abstract algebraic presentation has necessitated either virtually new proofs or radical changes in old ones. The idea of an algebraic treatment of the subject and certain aspects and implications of the algebraic development seem to be essentially new.

The desire to keep the work down to a reasonable size has affected both the matter and the form of the work. Certain material which seemed to be less interesting has been entirely omitted; problems less closely related to the main idea have been merely touched upon; the arguments throughout have been presented in a rather concise form; and counter-examples have not always been supplied.

I am greatly indebted to Professors Louise H. Chin (University of Arizona), Bjarni Jónsson (Brown University), and J. C. C. McKinsey (Oklahoma Agricultural and Mechanical College) for their help in preparing this work. The assistance of Miss Chin in preparing and revising the manuscript, reading proofs, and compiling the index was indeed invaluable. As will be seen from references in footnotes, Mr. Jónsson has enriched the work with a number of original contributions, and we have obtained jointly the results which are presented in the appendix.

It would be impossible for me to conclude this introduction without mentioning one more name—that of Adolf Lindenbaum, a former student and colleague of mine at the University of Warsaw. My close friend and collaborator for many years, he took a very active part in the earlier stages of the research which resulted in the present work, and the few references to his contributions that will be found in the book can hardly convey an adequate idea of the extent of my indebtedness. The wave of organized totalitarian barbarism engulfed this man of unusual intelligence and great talent—as it did millions of others.[4]

University of California,
Berkeley, July 1948

Alfred Tarski

[3] Footnotes to particular theorems will clear up the question of their origin. The results which are not referred to in footnotes have been obtained by the author and appear here—to his knowledge—for the first time. (This does not apply, of course, to various theorems of an elementary nature, like most of those which are stated in §1.)

[4] Adolf Lindenbaum was killed by the Gestapo in 1941.

PART I
ARITHMETIC OF CARDINAL ALGEBRAS

§1. POSTULATES AND ELEMENTARY CONSEQUENCES

We concern ourselves here with an algebraic system $\mathfrak{A}$ constituted by a SET A OF ARBITRARY ELEMENTS $a, b, c, \cdots$ and by two fundamental operations: the operation of FINITE (BINARY) ADDITION and that of INFINITE ADDITION. The first of these operations is performed on couples of elements $\langle a, b\rangle$, and the second on infinite sequences $\langle a_0, a_1, \cdots, a_i, \cdots\rangle$; the results are denoted by

$$a + b \quad \text{and} \sum_{i<\infty} a_i,$$

respectively. Thus, the algebra $\mathfrak{A}$ can be regarded as an ordered triple whose first term is the set A, and whose remaining two terms are the operations $+$ and $\sum$; in symbols,

$$\mathfrak{A} = \langle A, +, \textstyle\sum\rangle.$$

We shall not always adhere closely to the distinction between an algebra and the set of elements of this algebra; sometimes (especially in more informal remarks) we shall speak, e.g., of elements of $\mathfrak{A}$ meaning the elements of A.

The symbols

$$a \; \varepsilon \; A, \quad a, b \; \varepsilon \; A, \quad a_0, a_1, \cdots a_i, \cdots \; \varepsilon \; A$$

will express as usual the fact that elements a, or a and b, or $a_0, a_1, \cdots, a_i, \cdots$ belong to A.

DEFINITION 1.1. *An algebraic system $\mathfrak{A} = \langle A, +, \sum\rangle$ which satisfies the following postulates* I–VII *is called a* CARDINAL ALGEBRA, *for abbreviation, a C.A. (and under the same conditions the set A is said to be a C.A.* UNDER $+$ AND $\sum$*):*

I (FINITE CLOSURE POSTULATE). *If $a, b \; \varepsilon \; A$, then $a + b \; \varepsilon \; A$.*

II (INFINITE CLOSURE POSTULATE). *If $a_0, a_1, \cdots, a_i, \cdots \; \varepsilon \; A$, then*

$$\sum_{i<\infty} a_i \; \varepsilon \; A.$$

III (ASSOCIATIVE POSTULATE). *If $a_0, a_1, \cdots, a_i, \cdots \; \varepsilon \; A$, then*

$$\sum_{i<\infty} a_i = a_0 + \sum_{i<\infty} a_{i+1}.$$

IV (COMMUTATIVE-ASSOCIATIVE POSTULATE). *If* $a_0, a_1, \cdots, a_i, \cdots, b_0, b_1, \cdots, b_i, \cdots \varepsilon A$, *then*

$$\sum_{i<\infty} (a_i + b_i) = \sum_{i<\infty} a_i + \sum_{i<\infty} b_i.$$

V (POSTULATE OF THE ZERO ELEMENT). *There is an element* $z \varepsilon A$ *such that* $a + z = z + a = a$ *for every* $a \varepsilon A$.

VI (REFINEMENT POSTULATE). *If* $a, b, c_0, c_1, \cdots, c_i, \cdots \varepsilon A$ *and*

$$a + b = \sum_{i<\infty} c_i,$$

then there are elements $a_0, a_1, \cdots, a_i, \cdots, b_0, b_1, \cdots, b_i, \cdots \varepsilon A$ *such that*

$$a = \sum_{i<\infty} a_i, \quad b = \sum_{i<\infty} b_i, \quad and \quad c_n = a_n + b_n \quad for \quad n = 0, 1, 2, \cdots.$$

VII (REMAINDER, OR INFINITE CHAIN, POSTULATE). *If* $a_0, a_1, \cdots, a_i, \cdots, b_0, b_1, \cdots, b_i, \cdots \varepsilon A$ *and if* $a_n = b_n + a_{n+1}$ *for* $n = 0, 1, 2, \cdots$, *then there is an element* $c \varepsilon A$ *such that*

$$a_n = c + \sum_{i<\infty} b_{n+i} \quad for \quad n = 0, 1, 2, \cdots.$$

The elementary postulates I–V require no comments. It may be remarked only that these postulates are in themselves rather weak. To derive from our postulate system various elementary properties of cardinal algebras—like the ordinary commutative and associative laws for binary addition and the corresponding laws for infinite addition—we shall have to apply, besides I–V, also the remainder postulate VII; therefore, the first derivations in this section will take on a slightly artificial character (cf. the proof of Theorem 1.6 below).

The refinement postulate VI will lead us in §2 to two related theorems of a more symmetric nature: the general refinement theorem 2.1 and the finite refinement theorem 2.3; while in VI a binary sum is assumed to be equal to a sum of a sequence, the hypothesis of 2.1 involves two sums of sequences, and that of 2.3 two binary sums. Refinement theorems are known from the literature and have been applied many times in various algebraic discussions.[1] Theorem 2.1

[1] They occur often in the literature in a more general form involving systems of elements of an arbitrary power. Cf., for instance, Birkhoff [1], p. 286; Birkhoff [4], pp. 615 f.; Fitting [1], p. 380; Golowin [1], p. 424; Riesz [1], p. 175.

(even when restricted to infinite sequences) implies Postulate VI, and hence can equivalently replace the latter in our postulate system. This is not the case, however, so far as Theorem 2.3 is concerned; a counter-example will be found in §14. Nevertheless, Theorem 2.3 suffices for most derivations in the first part of this work. It may be noticed that the converse of Postulate VI is equivalent to Postulate IV.

Postulate VII seems not to be known in the literature. Its content is less clear than that of the remaining postulates. It expresses a certain minimal property of the sequence of REMAINDERS,

$$r_n = \sum_{i<\infty} b_{n+i},$$

of an infinite sum $\sum_{i<\infty} b_i$. By III, this sequence satisfies the condition

$$r_n = b_n + r_{n+1} \quad \text{for} \quad n = 0, 1, 2, \cdots;$$

whereas, by VII, every other sequence $a_0, a_1, \cdots, a_i, \cdots$ which satisfies the same condition can be obtained from the sequence $r_0, r_1, \cdots, r_i, \cdots$ by adding a constant term c. The converse of VII also holds in C.A.'s, and it implies III. The problem of replacing Postulate VII by simpler postulates deserves some attention. However, this postulate has certain properties which fully justify its inclusion in the postulate system. As we shall see, it has a great deductive power. The proof that the postulate holds in various special cardinal algebras is very simple; and this applies even to those algebras which are not of an elementary character, e.g., to the algebras of cardinal numbers and relation numbers which are discussed in §§17 and 18. Finally, as we shall see in §6 (cf. the proof of Theorem 6.10), Postulate VII has the property that, whenever it holds in a given algebra, it is also satisfied in a comprehensive class of coset algebras (homomorphic images) of this algebra. This property—which VII shares with other postulates of our system—is very important since the method of constructing coset algebras leads us to the most interesting examples of C.A.'s.

It may be noticed that Postulate VII is closely related to Theorem 2.21 (or 3.20) by which the sum of an infinite sequence is the least upper bound of its partial sums. If the hypothesis of VII is satisfied, then an application of 2.21 yields a conclusion which seems to be but slightly weaker than that of VII (in fact, the conclusion that, for

every integer n, there is an element c which satisfies the last formula of VII). At first sight, 2.21 could seem more natural and more suitable as a postulate than VII. Actually, however, 2.21 has none of the advantageous properties of VII which were mentioned above; e.g., a direct proof that 2.21 holds in various special C.A.'s—for instance, in the algebra of cardinal numbers—is by no means simple. Moreover, it can be shown by an example that 2.21 alone cannot equivalently replace VII in our postulate system. We shall return to this question in §16.

We now introduce in the arithmetic of C.A.'s the notions of the ZERO ELEMENT, 0; of the SUM OF A FINITE SEQUENCE $\langle a_0, a_1, \cdots, a_{n-1}\rangle$ OF ELEMENTS, $\sum_{i<n} a_i$; of the n^{th} MULTIPLE OF AN ELEMENT a, $n \cdot a$; and of the "LESS THAN OR EQUAL TO" RELATION, $\leqq$. All these notions are, of course, relativized to a given C.A., $\mathfrak{A} = \langle A, +, \sum\rangle$.

DEFINITION 1.2. *0 is the unique element $z \varepsilon A$ such that $a + z = z + a = a$ for every $a \varepsilon A$.*

DEFINITION 1.3. *For every integer n and for any elements a_0, $a_1, \cdots, a_i, \cdots \varepsilon A$ with $i < n$ we put*

$$\sum_{i<n} a_i = \sum_{i<\infty} b_i$$

where $b_i = a_i$ for $i < n$, and $b_i = 0$ for $i \geqq n$.

By an integer we always understand in this work a non-negative integer. In connection with Definition 1.3 it proves convenient henceforth to regard ∞ as a new, infinite number included in the system of integers. The usual rules for calculating with ∞ are assumed; thus, e.g.,

$$n + \infty = \infty + n = \infty,$$

$$0 \cdot \infty = \infty \cdot 0 = 0, \quad \text{and} \quad n \cdot \infty = \infty \cdot n = \infty \quad \text{for} \quad n \neq 0.$$

The formula

$$\text{(i)} \quad n < \infty, \quad \text{or} \quad \text{(ii)} \quad n \leqq \infty,$$

will be used to express the fact that n is (i) a finite, or (ii) an arbitrary, integer. The symbol

$$\sum_{i<n} a_i$$

has now a definite meaning both for $n < \infty$ and $n = \infty$; and we are able to state definitions and theorems applying to both cases simultaneously.

DEFINITION 1.4. *For every $n \leqq \infty$ and for every $a \varepsilon A$, we put $n \cdot a = \sum_{i<n} a$ (i.e., $n \cdot a = \sum_{i<n} b_i$ where $b_i = a$ for every $i < n$).*

DEFINITION 1.5. *We say that $a \leqq b$ (or $b \geqq a$) if $a, b \varepsilon A$ and if there is an element $c \varepsilon A$ with $a + c = b$.*

We shall sometimes write

$$a \leqq b \leqq c$$

instead of

$$a \leqq b \quad \text{and} \quad b \leqq c;$$

and similarly for other relation symbols.

From Postulates 1.1.I–VII and Definitions 1.2–1.5 we shall derive in this section various consequences of a rather elementary nature; thus, various COMMUTATIVE, ASSOCIATIVE, and DISTRIBUTIVE LAWS for sums and multiples; theorems stating fundamental properties of 0; theorems which show that the relation $\leqq$ establishes a partial order in a C.A.; and MONOTONY LAWS for the relation $\leqq$ (with respect to sums and multiples). When applying the closure postulates, we shall not refer to them explicitly. In formulating theorems, we shall usually omit the assumptions by which all the elements involved belong to a given C.A.

THEOREM 1.6. *$0 \varepsilon A$, and $a + 0 = 0 + a = a$ for every a.*
PROOF: by 1.1.V and 1.2.

THEOREM 1.7. *$n \cdot 0 = 0$ for every $n \leqq \infty$.*
PROOF: By 1.1.VII (with $a_i = b_i = 0$ for $i = 0, 1, 2, \cdots$), 1.4, and 1.6 we obtain for some c

$$c + \infty \cdot 0 = 0,$$

and hence, by 1.1.III,

$$\sum_{i<\infty} c_i = 0 \quad \text{where} \quad c_0 = c \quad \text{and} \quad c_i = 0 \quad \text{for} \quad i = 1, 2, \cdots.$$

Consequently, by 1.4, 1.6, and 1.1.IV,

$$\infty \cdot 0 = 0 + \infty \cdot 0 = \sum_{i<\infty} c_i + \infty \cdot 0 = \sum_{i<\infty} (c_i + 0) = \sum_{i<\infty} c_i = 0;$$

and hence, by 1.3 and 1.4, we arrive at the conclusion.

THEOREM 1.8. *$\sum_{i<0} a_i = 0$, $\sum_{i<1} a_i = a_0$, and $\sum_{i<2} a_i = a_0 + a_1$.*

PROOF: by 1.1.III, 1.3, 1.6, and 1.7.

We note that Postulate 1.1.I is not involved in the proofs of 1.6–1.8. Hence we conclude, in view of the last formula of 1.8, that this postulate can be formally derived from the remaining postulates (1.1.II–V and VII).

COROLLARY 1.9. *$0 \cdot a = 0$, $1 \cdot a = a$, and $2 \cdot a = a + a$.*

PROOF: by 1.4 and 1.8.

THEOREM 1.10. *$a = b + c$ if, and only if, there are elements d_0, $d_1, \cdots, d_i, \cdots$ with*

$$a = \sum_{i<\infty} d_i, \quad b = d_0, \quad \textit{and} \quad c = \sum_{i<\infty} d_{i+1}.$$

PROOF: by 1.1.III, 1.3, and 1.8.

THEOREM 1.11. *$\sum_{i<n} (a_i + b_i) = \sum_{i<n} a_i + \sum_{i<n} b_i$ for every $n \leqq \infty$.*

PROOF: by 1.1.IV, 1.3, and 1.6.

COROLLARY 1.12. *$n \cdot (a + b) = n \cdot a + n \cdot b$ for every $n \leqq \infty$.*

PROOF: by 1.4 and 1.11.

THEOREM 1.13. *$a + b = b + a$.*

PROOF: by 1.6, 1.8, and 1.11 (with $n = 2$, $a_0 = b_1 = 0$, $a_1 = b$, and $b_0 = a$).

THEOREM 1.14. *$(a + b) + c = a + (b + c)$.*

PROOF: by 1.6, 1.8, and 1.11 (with $n = 2$, $a_0 = a$, $a_1 = 0$, $b_0 = b$, and $b_1 = c$).

Henceforth, we shall omit parentheses in expressions of the type '$(a + b) + c$' or '$a + (b + c)$'.

THEOREM 1.15. *$\sum_{i<n+1} a_i = a_0 + \sum_{i<n} a_{i+1}$ for every $n \leqq \infty$.*

PROOF: by 1.1.III and 1.3.

COROLLARY 1.16. $(n+1)\cdot a = a + n\cdot a = n\cdot a + a$ *for every* $n \leqq \infty$; *in particular*, $\infty \cdot a = a + \infty \cdot a$.
PROOF: by 1.4, 1.13, and 1.15.

THEOREM 1.17. $\sum_{i<n+1} a_i = \sum_{i<n} a_i + a_n$ *for every* $n < \infty$.
PROOF: by induction based on 1.6, 1.8, 1.14, and 1.15.

From 1.8 and 1.17 (or 1.15) it is seen that sums of finite sequences can be defined in the usual recursive way in terms of binary addition.

THEOREM 1.18. *If* $n < \infty$ *and* $p \leqq \infty$, *then*

$$\sum_{i<n} a_i + \sum_{i<p} a_{i+n} = \sum_{i<n+p} a_i .$$

PROOF: by induction with respect to n, and with the help of 1.6, 1.8, 1.14, 1.15, and 1.17.

THEOREM 1.19. $a \leqq a$ *and* $0 \leqq a$.
PROOF: by 1.5 and 1.6.

THEOREM 1.20. *If* $a \leqq b$ *and* $b \leqq c$, *then* $a \leqq c$.
PROOF: by 1.5 and 1.14.

THEOREM 1.21. *If* $a \leqq b$, *then* $a + c \leqq b + c$.
PROOF: by 1.5, 1.13, and 1.14.

THEOREM 1.22. *If* $n \leqq \infty$ *and* $a_i \leqq b_i$ *for every* $i < n$, *then*

$$\sum_{i<n} a_i \leqq \sum_{i<n} b_i .$$

PROOF: by 1.5 and 1.11.

COROLLARY 1.23. *If* $n \leqq \infty$ *and* $a \leqq b$, *then* $n\cdot a \leqq n\cdot b$.
PROOF: by 1.4 and 1.22.

THEOREM 1.24. *If* $n \leqq p \leqq \infty$, *then*

$$\sum_{i<n} a_i \leqq \sum_{i<p} a_i .$$

PROOF: by 1.5, 1.18, and 1.19.

COROLLARY 1.25. *If* $n \leqq p \leqq \infty$, *then* $n\cdot a \leqq p\cdot a$.
PROOF: by 1.4 and 1.24.

THEOREM 1.26. *If $m < n \leqq \infty$, then*

$$a_m \leqq \sum_{i<n} a_i .$$

PROOF: by 1.5, 1.20, 1.21, and 1.24.

COROLLARY 1.27. *If $0 < n \leqq \infty$, then $a \leqq n \cdot a$.*
PROOF: by 1.4 and 1.26.

THEOREM 1.28. *If $a + b = b$ and $b \leqq c$, then $a + c = c$.*
PROOF: by 1.5 and 1.14.

Formulas of the type '$a + b = b$' (which occur in 1.28) can be read 'a IS ABSORBED BY b' or 'b ABSORBS a'. The relation of absorption plays an important part in the arithmetic of C.A.'s.

THEOREM 1.29. *In order that $a + b = b$, it is necessary and sufficient that $\infty \cdot a \leqq b$.*
PROOF: The necessity of the condition follows from 1.1.VII (with $a_i = b$ and $b_i = a$ for $i = 0, 1, 2, \cdots$), 1.4, 1.5, and 1.13; the sufficiency follows from 1.16 (for $n = \infty$) and 1.28.

THEOREM 1.30. *If $a \leqq b$ and $b + c = c$, then $a + c = c$.*
PROOF: By 1.23 and 1.29 we have

$$\infty \cdot a \leqq \infty \cdot b \leqq c.$$

Hence, by applying 1.20 and again 1.29, we obtain the conclusion.

THEOREM 1.31.[2] *If $a \leqq b$ and $b \leqq a$, then $a = b$.*
PROOF: By 1.5 and 1.13 we have, for some c and d,

(1) $$c + a = b, \quad d + b = a,$$

and further, with the help of 1.14,

$$(d + c) + a = a \quad \text{and} \quad c \leqq d + c.$$

Hence, by 1.3C,

[2] When applied to cardinal numbers, 1.31 coincides with the Cantor-Bernstein equivalence theorem; cf. Bernstein [1], p. 120. The proof of Theorem 1.31 (and of Theorems 1.29 and 1.30 upon which it rests) is essentially an algebraic reconstruction of the proof of the equivalence theorem given in Zermelo [1]. For other proofs of the equivalence theorem see Schoenflies [1], pp. 34 ff.

(2) $$c + a = a.$$

The conclusion is implied by (1) and (2).

Theorems 1.19, 1.20, and 1.31 show that the relation $\leqq$ is REFLEXIVE, TRANSITIVE, and ANTISYMMETRIC in the set A; consequently, the set A is PARTIALLY ORDERED by this relation (and, of course also by the converse relation $\geqq$).

COROLLARY 1.32. *If $a \leqq 0$, then $a = 0$.*
PROOF: By 1.19 and 1.31.

COROLLARY 1.33. *If $a + b = 0$, then $a = b = 0$.*
PROOF: by 1.5, 1.13, and 1.32.

COROLLARY 1.34. *If $m < n \leqq \infty$ and*

$$\sum_{i<n} a_i = 0,$$

then $a_m = 0$.
PROOF: by 1.26 and 1.32.

COROLLARY 1.35. *If $0 < n \leqq \infty$ and $n \cdot a = 0$, then $a = 0$.*
PROOF: by 1.27 and 1.32.

THEOREM 1.36. *$a = \sum_{i<\infty} b_i$ if, and only if, the following conditions are satisfied:*

(i) *there is an infinite sequence of elements $a_0, a_1, \cdots, a_i, \cdots$ with $a_0 = a$ and $a_n = b_n + a_{n+1}$ for $n = 0, 1, 2, \cdots$;*

(ii) *if, for any other sequence $c_0, c_1, \cdots, c_i, \cdots$ we have $c_n = b_n + c_{n+1}$ for $n = 0, 1, 2, \cdots$, then $a \leqq c_0$.*

PROOF: by 1.1.III, 1.1.VII, 1.5, 1.13, and 1.31.

From 1.10 and 1.36 it is seen that each of the two fundamental operations, $+$ and $\sum$, of a C.A. can be defined by means of the other. Hence C.A.'s can be characterized as algebraic systems with one operation only—either with the binary operation $+$ or with the infinite operation $\sum$. This, however, would hardly contribute to a simplification of the postulate system. If we wish to characterize C.A.'s as algebras with the unique operation $\sum$, we can use, e.g., the following system of postulates:

I (CLOSURE POSTULATE). *Postulate* 1.1.II.

II (COMMUTATIVE-ASSOCIATIVE POSTULATE). *Theorem* 1.44 *below with $n = p = \infty$.*

III (POSTULATE OF THE ZERO ELEMENT). *There is an element* $z \varepsilon A$ *such that, for any elements* $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$,

(i) $$\sum_{i<\infty} a_i = \sum_{i<\infty} a_{i+1} \quad \textit{if} \quad a_0 = z;$$

(ii) $$\sum_{i<\infty} a_i = a_0 \quad \textit{if} \quad a_{i+1} = z \quad \textit{for} \quad i = 0, 1, 2, \cdots.$$

IV (REFINEMENT POSTULATE). *Theorem* 2.1 *below with* $n = p = \infty$.

V (REMAINDER POSTULATE). *If* $a_{i,j} \varepsilon A$ *for* $i, j = 0, 1, 2, \cdots$, *and*

$$\sum_{j<\infty} a_{n,j+1} = \sum_{j<\infty} a_{n+1,j} \quad \textit{for every} \quad n < \infty,$$

then there is an element $c \varepsilon A$ *such that, for any given* $n < \infty$,

$$\sum_{j<\infty} a_{n,j+1} = \sum_{i<\infty} c_i$$

where $c_0 = c$ *and* $c_{i+1} = a_{n+i+1,0}$ *for* $i = 0, 1, 2, \cdots$.

THEOREM 1.37. $\infty \cdot a = b$ *if, and only if,*

(i) $$a + b = b,$$

and

(ii) $$\textit{for every} \quad c, \quad a + c = c \quad \textit{implies} \quad b \leqq c.$$

PROOF: by 1.16, 1.29, and 1.31.

By saying that two sequences $a_0, a_1, \cdots, a_i, \cdots$ with $i < n \leqq \infty$ and $b_0, b_1, \cdots, b_j, \cdots$ with $j < p \leqq \infty$ differ at most in their order, we shall of course mean that there is a function f which maps in a one-to-one way the set of all integers $i < n$ onto the set of all integers $j < p$ and which satisfies the condition

$$a_i = b_{f(i)} \quad \text{for} \quad i < n;$$

in this case the numbers n and p must obviously be equal. In an analogous sense we say that an element a has, apart from order, only one representation

$$a = \sum_{i<n} a_i$$

where the elements a_i are subjected to certain conditions. We shall use similar terminology also in case the elements a_i and possibly b_j

are correlated, not with numbers $i < n$ and $j < p$, but with elements i and j of two arbitrary sets.

THEOREM 1.38. *If $n \leqq \infty$ and if the sequences $a_0, a_1, \cdots, a_i, \cdots$ and $b_0, b_1, \cdots, b_i, \cdots$ with $i < n$ differ at most in their order, then*

$$\sum_{i<n} a_i = \sum_{i<n} b_i.$$

PROOF: If $n < \infty$ we apply a familiar inductive procedure (based upon 1.8, 1.13–1.15, and 1.18). If $n = \infty$, we construct a sequence $c_0, c_1, \cdots, c_i, \cdots$ such that

$$c_0 = \sum_{i<\infty} a_i \quad \text{and} \quad c_m = b_m + c_{m+1} \quad \text{for} \quad m = 0, 1, 2, \cdots$$

(c_m is the sum obtained from $\sum_{i<\infty} a_i$ by omitting the terms equal to $b_0, b_1, \cdots, b_i, \cdots$ with $i < m$). By 1.36 we have

$$\sum_{i<\infty} b_i \leqq c_0 = \sum_{i<\infty} a_i.$$

In a similar way we obtain the inequality in the opposite direction, and, hence, by 1.31, the desired equation.

THEOREM 1.39. *If $n \leqq \infty$, then*

$$\sum_{i<2\cdot n} a_i = \sum_{i<n} a_{2\cdot i} + \sum_{i<n} a_{2\cdot i+1}.$$

PROOF: By 1.11, it suffices to show that

$$\sum_{i<2\cdot n} a_i = \sum_{i<n} (a_{2\cdot i} + a_{2\cdot i+1}).$$

The proof of this formula is analogous to that of 1.38.

THEOREM 1.40. *Let n and p be $\leqq \infty$; and let $k_0, k_1, \cdots, k_i, \cdots$ with $i < n + p$, $l_0, l_1, \cdots, l_i, \cdots$ with $i < n$, and $m_0, m_1, \cdots, m_i, \cdots$ with $i < p$ be three sequences of finite non-negative integers, without repeating terms, such that every term of the first sequence occurs in one and only one of the remaining two sequences, and conversely. Then*

$$\sum_{i<n+p} a_{k_i} = \sum_{i<n} a_{l_i} + \sum_{i<p} a_{m_i}.$$

PROOF: In case $n = p$ the conclusion follows easily from 1.38 and 1.39; in case $n < p$, or $p < n$, we have to apply in addition 1.13, 1.14, and 1.18.

COROLLARY 1.41. *If* $n \leqq \infty$ *and* $p \leqq \infty$, *then* $(n + p)\cdot a = n\cdot a + p\cdot a$; *in particular,* $\infty\cdot a = n\cdot a + \infty\cdot a = \infty\cdot a + \infty\cdot a$.

PROOF: by 1.4 and 1.40.

THEOREM 1.42. *Let* $n, q, l_0, l_1, \cdots, l_i, \cdots$ *be* $\leqq \infty$. *If a function* f *maps in a one-to-one way the set of all couples* $\langle i, j\rangle$ *with* $i < n$ *and* $j < l_i$ *for* $i < n$ *onto the set of all numbers* $k < q$, *and if* $a_{i,j} = b_{f(i,j)}$ *for all* $i < n$ *and* $j < l_i$, *then*

$$\sum_{i<n} \sum_{j<l_i} a_{i,j} = \sum_{k<q} b_k .$$

PROOF: In case $n < \infty$ we proceed by an easy induction based upon 1.8, 1.17, and 1.40. In case $n = \infty$ we apply the same method as in the proof of 1.38.

COROLLARY 1.43. *If* $n \leqq \infty$ *and* $p \leqq \infty$, *then* $n\cdot(p\cdot a) = (n\cdot p)\cdot a = p\cdot(n\cdot a)$; *in particular, if* $n \neq 0$, *then* $\infty\cdot a = n\cdot(\infty\cdot a) = \infty\cdot(n\cdot a) = \infty\cdot(\infty\cdot a)$.

PROOF: We put in 1.42

$$q = n\cdot p, \quad l_i = p, \quad \text{and} \quad a_{i,j} = b_k = a \quad \text{for} \quad i < n, \quad j < p, \quad \text{and} \quad k < q.$$

A function f satisfying the hypothesis of 1.42 can easily be constructed. The conclusion follows by 1.4.

THEOREM 1.44. *If* $n \leqq \infty$ *and* $p \leqq \infty$, *then*

$$\sum_{i<n} \sum_{j<p} a_{i,j} = \sum_{j<p} \sum_{i<n} a_{i,j} .$$

PROOF: We construct the function f which satisfies the hypothesis of 1.42 for $q = n\cdot p$ and $l_i = p$; then we apply 1.42 twice—the first time with the function f, and the second time with the converse function g, $g(j, i) = f(i, j)$ for $i < n$ and $j < p$.

COROLLARY 1.45. *If* $n \leqq \infty$ *and* $p \leqq \infty$, *then*

$$\sum_{i<n} (p\cdot a_i) = p\cdot \sum_{i<n} a_i .$$

PROOF: by 1.5 and 1.44.

THEOREM 1.46.[3] *If* $0 < n \leqq \infty$, *then, in order that* $n\cdot a + b = b$, *it is necessary and sufficient that* $a + b = b$.

[3] Theorems 1.46 and 1.47 in their applications to cardinal numbers are due to E. Zermelo; cf. Zermelo [1].

PROOF: By 1.5, 1.13, 1.21, and 1.27, we have

$$b \leqq a + b \leqq n \cdot a + b;$$

hence the necessity of the condition follows from 1.31. On the other hand, by 1.29 and 1.43 (with $n = \infty$ and $p = n$), the formula

$$a + b = b$$

implies

$$\infty \cdot (n \cdot a) = \infty \cdot a \leqq b,$$

and hence, again by 1.29,

$$n \cdot a + b = b.$$

Thus, the condition of the theorem is also sufficient.

THEOREM 1.47 (ABSORPTION THEOREM). *If $n \leqq \infty$, then, in order that*

$$\sum_{i<n} a_i + b = b,$$

it is necessary and sufficient that $a_i + b = b$ for every $i < n$.

PROOF: The proof of necessity is the same as in 1.46, with 1.27 replaced by 1.26. In case $n < \infty$, we show that the condition is sufficient by an easy induction (1.6, 1.8, 1.14, and 1.17). If, finally,

$$a_i + b = b \quad \text{for} \quad i < \infty,$$

we have by 1.46

$$\infty \cdot a_i + b = b \quad \text{for} \quad i < \infty.$$

Hence, by 1.1.VII (with $a_i = b$ and $b_i = \infty \cdot a_i$), 1.5, 1.13, and 1.45,

$$\infty \cdot (\sum_{i<\infty} a_i) = \sum_{i<\infty} (\infty \cdot a_i) \leqq b;$$

and finally, by 1.29,

$$\sum_{i<\infty} a_i + b = b.$$

Thus, the condition in question is sufficient also in case $n = \infty$.

In our further discussion we shall apply freely Postulates 1.1.I–V and the theorems derived in this section, without referring to them explicitly (except for the less familiar laws concerning the relation of absorption, like 1.29 or 1.47).

§ 2. FUNDAMENTAL THEOREMS

We turn now to less elementary results of the arithmetic of C.A.'s.[4] We shall make an essential use here of the refinement postulate, 1.1.VI, which was not applied in §1 at all.

THEOREM 2.1 (GENERAL REFINEMENT THEOREM).[5] *If* $n \leqq \infty$, $p \leqq \infty$, *and*

$$\sum_{i<n} a_i = \sum_{j<p} b_j ,$$

then there is a double sequence of elements $c_{i,j}$ *such that*

$$a_i = \sum_{j<p} c_{i,j} \quad \textit{for} \quad i < n, \quad \textit{and} \quad b_j = \sum_{i<n} c_{i,j} \quad \textit{for} \quad j < p.$$

PROOF: Consider first the case when $n = p = \infty$. By 1.1.III, the infinite sequences $a_0, a_1, \cdots, a_i, \cdots$ and $b_0, b_1, \cdots, b_i, \cdots$ satisfy the formula

$$a_0 + \sum_{i<\infty} a_{i+1} = \sum_{i<\infty} b_i .$$

Hence, by putting in 1.1.VI,

$$a = a_0, \quad b = \sum_{i<\infty} a_{i+1}, \quad \text{and} \quad c_i = b_i \quad \text{for} \quad i = 0, 1, 2, \cdots ,$$

we obtain two new sequences $d_{0,0}, d_{1,0}, \cdots, d_{i,0}, \cdots$ and $e_{0,0}, e_{1,0}, \cdots, e_{i,0}, \cdots$ which satisfy the conclusion of 1.1.VI. We now have

$$e_{0,0} + \sum_{i<\infty} e_{i+1,0} = \sum_{i<\infty} a_{i+1} ;$$

and, by applying 1.1.VI again, we obtain two further sequences, $d_{i,1}$, and $e_{i,1}$. Assuming that, for any given $n < \infty$, the sequences $d_{i,n}$, $e_{i,n}$, $d_{i,n+1}$, and $e_{i,n+1}$ have been defined and that

$$e_{0,n+1} + \sum_{i<\infty} e_{i+1,n+1} = \sum_{i<\infty} e_{i+1,n} ,$$

[4] Several theorems of this section—2.6, 2.8, 2.10, 2.12–2.15, 2.27, 2.30, 2.33, 2.34, and 2.37—are stated for cardinal numbers without proof (and sometimes in a weaker form) in Lindenbaum-Tarski [1], pp. 301 ff. Regarding the origin of 2.10, 2.33, 2.34, and 2.37, see footnotes below; the remaining theorems originate with the author.

[5] In connection with 2.1 and 2.3 cf. footnote 1 in §1. A direct proof of these theorems for cardinal numbers is obviously very elementary.

we apply the same procedure to the last formula, and we arrive at the sequences $d_{i,n+2}$ and $e_{i,n+2}$. We show by induction that

$$a_k = \sum_{j<k} d_{k-j-1,2\cdot j+1} + \sum_{j<\infty} d_{j,2\cdot k} \quad \text{for} \quad k < \infty,$$

and

$$b_l = \sum_{i<l+1} d_{l-i,2\cdot i} + \sum_{i<\infty} d_{i,2\cdot l+1} \quad \text{for} \quad l < \infty.$$

Hence, the double sequence of elements $c_{i,j}$, defined by the formulas

$$c_{i,j} = d_{j-i,2\cdot i} \quad \text{for} \quad i \leqq j < \infty$$

and

$$c_{i,j} = d_{i-j-1,2\cdot j+1} \quad \text{for} \quad j < i < \infty,$$

satisfies the conclusion. The general case reduces easily to the one just considered; we extend the sequences $a_0, a_1, \cdots, a_i, \cdots$ and $b_0, b_1, \cdots, b_i, \cdots$ to infinite sequences by putting

$$a_i = b_j = 0 \quad \text{for} \quad n \leqq i < \infty \quad \text{and} \quad p \leqq j < \infty.$$

The METHOD OF ITERATION, which was employed for the first time in the proof of 2.1, is very characteristic for the theory of C.A.'s (cf., for instance, the proofs of 2.6, 2.31, and 6.10 below).

COROLLARY 2.2. *If $n \leqq \infty$ and*

$$a \leqq \sum_{i<n} b_i,$$

then there are elements $a_0, a_1, \cdots, a_i, \cdots$ such that

$$a = \sum_{i<n} a_i, \quad \textit{and} \quad a_i \leqq b_i \quad \text{for} \quad i < n.$$

PROOF: by 2.1 with $n = 2$ and $p = n$.

THEOREM 2.3 (FINITE REFINEMENT THEOREM). *If $a_1 + a_2 = b_1 + b_2$, then there are elements c_1, c_2, c_3, and c_4 for which $a_1 = c_1 + c_2$, $a_2 = c_3 + c_4$, $b_1 = c_1 + c_3$, and $b_2 = c_2 + c_4$.*

PROOF: by 2.1 for $n = p = 2$.

COROLLARY 2.4. *If $a \leqq b_1 + b_2$, then there are elements a_1 and a for which $a = a_1 + a_2$, $a_1 \leqq b_1$, and $a_2 \leqq b_2$.*

PROOF: by 2.3.

COROLLARY 2.5. *If* $a + b = 2 \cdot c$, *then there are elements* a', b', *and* c' *for which* $a = 2 \cdot a' + c'$, $b = 2 \cdot b' + c'$, *and* $c = a' + b' + c'$.

PROOF: By applying 2.3 to the hypothesis, we obtain

$$a = d_1 + d_2, \quad b = d_3 + d_4, \quad \text{and} \quad c = d_1 + d_3 = d_2 + d_4.$$

By applying now 2.3 to the last of these formulas, we easily arrive at the conclusion.

Corollary 2.5 will serve as a lemma in the proof of 2.31.

The following theorem 2.6 is closely related to 2.3 and has also the character of a refinement theorem; its proof, however, is less elementary, for it contains an infinite construction and requires a combined application of the refinement postulate and the remainder postulate.

THEOREM 2.6. *If* $a + c = b + c$, *then there are elements* a', b', *and* d *such that* $a = a' + d$, $b = b' + d$, *and* $c = a' + c = b' + c$.

PROOF: We put

$$a_0 = a, \quad b_0 = b, \quad \text{and} \quad c_0 = c.$$

Assuming that, for any given $n < \infty$, the elements a_n, b_n, and c_n with

$$a_n + c_n = b_n + c_n$$

have been defined, we apply 2.3 and obtain four elements a_{n+1}, b_{n+1}, c_{n+1}, and d_n for which

$$(1) \qquad a_n = a_{n+1} + d_n, \quad b_n = b_{n+1} + d_n,$$

and

$$(2) \qquad c_n = a_{n+1} + c_{n+1} = b_{n+1} + c_{n+1};$$

in view of (2), the procedure can be continued indefinitely. By applying 1.1.VII to (1), we get elements a' and b' with

$$(3) \qquad a_n = a' + \sum_{i<\infty} d_{n+i} \quad \text{and} \quad b_n = b' + \sum_{i<\infty} d_{n+i} \qquad \text{for} \quad n = 0, 1, 2, \cdots.$$

Hence, by putting

$$d = \sum_{i<\infty} d_i,$$

we obtain for $n = 0$

(4) $$a = a' + d \quad \text{and} \quad b = b' + d.$$

Moreover, (3) implies

(5) $$\infty \cdot a' \leqq \sum_{i<\infty} a_{i+1} \quad \text{and} \quad \infty \cdot b' \leqq \sum_{i<\infty} b_{i+1};$$

while (2) gives by 1.1.VII

(6) $$\sum_{i<\infty} a_{i+1} \leqq c_0 = c \quad \text{and} \quad \sum_{i<\infty} b_{i+1} \leqq c_0 = c.$$

From (5) and (6) we obtain by 1.29

(7) $$c = a' + c = b' + c.$$

In view of (4) and (7), the proof is complete.

Corollary 2.7. *If $a + c \leqq b + c$, then there are elements a_1 and a_2 for which $a = a_1 + a_2$, $a_1 \leqq b$, and $c = a_2 + c$.*

Proof: by 2.6, with the help of 1.30 and 2.4.

Theorem 2.8. *If $a + c = b + c$, then there is an element c' such that $a + c' = b + c'$ and $c = c' + c$.*

Proof: We apply 2.6 and put

$$c' = \infty \cdot a' + \infty \cdot b'.$$

The conclusion follows by 1.16 (with $n = \infty$) and 1.46.

Corollary 2.9. *If $a + c \leqq b + c$, then there is an element c' for which $a + c' \leqq b + c'$ and $c = c' + c$.*

Proof: by 2.8.

Theorems 2.6 and 2.8 are important for the reason that the cancellation law for sums fails in C.A.'s; i.e., $a + c = b + c$ does not always imply $a = b$. (We have, e.g., $a + \infty \cdot a = 0 + \infty \cdot a = \infty \cdot a$ even if $a \neq 0$.) Theorems 2.6 and 2.8 express, however, the fact that, if $a + c = b + c$, then a and b are equal up to certain elements which are absorbed by c; hence they can be successfully applied on various occasions instead of the lacking cancellation law. Similar remarks apply to 2.7 and 2.9 in connection with the cancellation law for sums in inequalities.

Certain special cases of cancellation laws for sums, which hold in the arithmetic of C.A.'s, are stated in 2.10–2.15.

THEOREM 2.10.[6] *If $n < \infty$ and $a + n \cdot c \leqq b + (n + 1) \cdot c$, then $a \leqq b + c$.*

PROOF: For $n = 0$ the theorem holds trivially; for $n = 1$ it follows from 2.9 (with 'b' changed to '$b + c$'); and in the general case it can be obtained by induction.

COROLLARY 2.11. *If $n < \infty$ and $a + n \cdot c \leqq b + n \cdot c$, then $a + c \leqq b + c$.*

PROOF: obvious in case $n = 0$; by 2.10 in case $n \neq 0$.

COROLLARY 2.12. *If $n < \infty$ and $a + n \cdot c = b + n \cdot c$, then $a + c = b + c$.*

PROOF: by 2.11.

Notice the difference between 2.10 and 2.12; the symbol '$\leqq$' in 2.10 cannot be replaced by '$=$'.

COROLLARY 2.13. *If $a + c \leqq b + c$ and $c \leqq b$, then $a \leqq b$.*
PROOF: by 2.10 (with $n = 1$).

COROLLARY 2.14. *If $a + c \leqq b + c$ and $b \leqq c$, then $a \leqq c$.*
PROOF: by 2.10 (with $n = 1$).

COROLLARY 2.15. *If $a + c = b + c$, $c \leqq a$, and $c \leqq b$, then $a = b$.*
PROOF: by 2.13.

THEOREM 2.16. *If $n \leqq \infty$ and $a_i + c = b_i + c$ for all $i < n$, then*

(i) $$\sum_{i<n} a_i + c = \sum_{i<n} b_i + c;$$

(ii) *in case $n \neq 0$, there are elements $c_0, c_1, \cdots, c_i, \cdots$ such that*

$$c = \sum_{i<n} c_i,$$

and $a_i + c_i = b_i + c_i$ for every $i < n$.

PROOF: We can obviously assume that $n \neq 0$. We put

$$c_0 = c.$$

By 2.8 we obtain elements c_i with

$$a_i + c_i = b_i + c_i \quad \text{and} \quad c = c_i + c \quad \text{for} \quad 0 < i < n.$$

[6] A special case of Theorem 2.10 (from which, however, the theorem can easily be derived) was established for cardinal numbers in Bernstein [1], p. 127.

Hence, by 1.47,

$$c = \sum_{i<n} c_i.$$

The conclusion follows easily.

Corollary 2.17. *If $n \leqq \infty$, and $a_i + c \leqq b_i + c$ for all $i < n$, then*

$$\sum_{i<n} a_i + c \leqq \sum_{i<n} b_i + c.$$

Proof: by 2.16.

It may be noticed that, in case $n < \infty$, Theorem 2.16(i) and Corollary 2.17 can be proved by an elementary induction, without the use of 2.9.

Theorem 2.18. *If $0 < n \leqq \infty$ and*

$$\sum_{i<n} a_i + c = b + c,$$

then

(i) *there are elements $b_0, b_1, \cdots, b_i, \cdots$ such that*

$$b = \sum_{i<n} b_i,$$

and $a_i + c = b_i + c$ for every $i < n$;

(ii) *there are also elements $c_0, c_1, \cdots, c_i, \cdots$ such that*

$$c = \sum_{i<n} c_i,$$

and $a_i + c_i = b_i + c_i$ for every $i < n$.

Proof: By 2.6 we have

$$\sum_{i<n} a_i = a' + d, \quad b = b' + d, \quad \text{and} \quad c = a' + c = b' + c.$$

By applying 2.1 (with $p = 2$) to the first of these formulas, we obtain elements a_i' and d_i with

$$a_i = a_i' + d_i \quad \text{for} \quad i < n, \quad a' = \sum_{i<n} a_i', \quad \text{and} \quad d = \sum_{i<n} d_i.$$

We now put

$$b_0 = b' + d_0, \quad \text{and} \quad b_i = d_i \quad \text{for} \quad 0 < i < n.$$

An elementary argument shows that the elements b_i thus defined satisfy the first part of the conclusion (1.28 is useful here). The second part follows by 2.16.

THEOREM 2.19. *If* $n < \infty$ *and*

$$a + \sum_{i<n} b_i = \sum_{i<n} b_i,$$

then there are elements $a_0, a_1, \cdots, a_i, \cdots$ *such that*

$$a = \sum_{i<n} a_i, \quad \text{and} \quad a_i + b_i = b_i \quad \text{for all } i < n.$$

PROOF: The theorem obviously holds for $n = 0, 1$. If $n = 2$, we have

$$(a + b_1) + b_0 = b_1 + b_0 ;$$

hence, by 2.8,

(1) $$(a + c_0) + b_1 = c_0 + b_1 \quad \text{with} \quad c_0 + b_0 = b_0 ;$$

and, again by 2.8,

(2) $$a + c_0 + c_1 = c_0 + c_1 \quad \text{with} \quad c_1 + b_1 = b_1 .$$

Thus,

$$a \leqq c_0 + c_1 ;$$

therefore, by 2.4,

(3) $$a_0 + a_1 = a, \quad a_0 \leqq c_0, \quad \text{and} \quad a_1 \leqq c_1 ;$$

and, by 1.30, we obtain from (1)–(3)

(4) $$a_0 + b_0 = b_0 \quad \text{and} \quad a_1 + b_1 = b_1 .$$

By (3) and (4), the elements a_0 and a_1 satisfy the conclusion. The result can be extended by a simple induction to an arbitrary $n < \infty$.

THEOREM 2.20. *If* $0 < n < \infty$ *and*

$$a + \sum_{i<n} c_i = b + \sum_{i<n} c_i,$$

then there are elements $a_0, a_1, \cdots, a_i, \cdots$ *and* $b_0, b_1, \cdots, b_i, \cdots$ *such that*

$$a = \sum_{i<n} a_i, \quad b = \sum_{i<n} b_i,$$

and $a_i + c_i = b_i + c_i$ *for every* $i < n$.

PROOF: By 2.6 we have

$$a = a' + d, \quad b = b' + d, \quad \text{and} \quad \sum_{i<n} c_i = a' + \sum_{i<n} c_i = b' + \sum_{i<n} c_i.$$

From the last of these formulas we obtain by 2.19

$$a' = \sum_{i<n} a_i', \quad b' = \sum_{i<n} b_i', \quad \text{and} \quad c_i = a_i' + c_i = b_i' + c_i \quad \text{for} \quad i < n.$$

We put

$$a_0 = a_0' + d, \quad b_0 = b_0' + d, \quad \text{and} \quad a_i = a_i', \quad b_i = b_i' \quad \text{for} \quad 0 < i < n.$$

It is easily seen that the elements a_i and b_i satisfy the conclusion.

Theorems 2.19 and 2.20 cannot be extended to the case when $n = \infty$. Theorems 2.18 and 2.20—like 2.1, 2.3, and 2.6—have the character of refinement theorems. It is possible to formulate a general theorem which comprehends 2.1, 2.18, and 2.20 as particular cases, and which establishes a decomposition of elements satisfying the formula

$$\sum_{i<n} a_i + \sum_{k<q} c_k = \sum_{j<p} b_j + \sum_{k<q} c_k$$

$$(0 < n \leqq \infty, 0 < p \leqq \infty, 0 < q < \infty).$$

This theorem will not be stated here.

THEOREM 2.21 (FUNDAMENTAL LAW OF INFINITE ADDITION).[7] *If*

$$\sum_{i<n} a_i \leqq b \quad \textit{for every} \quad n < \infty,$$

then

$$\sum_{i<\infty} a_i \leqq b.$$

PROOF: By hypothesis, there are elements $c_0, c_1, \cdots, c_i, \cdots$ such that

$$c_n + \sum_{i<n} a_i = b, \tag{1}$$

and therefore

$$c_n + \sum_{i<n} a_i = (c_{n+1} + a_n) + \sum_{i<n} a_i \quad \text{for every} \quad n < \infty.$$

[7] Theorem 2.21 is well known from the arithmetic of cardinals; its earlier proofs, however, involved essentially the well-ordering principle. Compare, e.g., Schoenflies [1], p. 135.

Hence, by 2.6, we obtain

(2) $$c_n = a'_n + d_n, \quad c_{n+1} + a_n = b'_n + d_n$$

and

$$\sum_{i<n} a_i = a'_n + \sum_{i<n} a_i = b'_n + \sum_{i<n} a_i \leqq b.$$

By 1.28, the latter formula gives

$$b'_n + b = b,$$

and hence, by 1.47,

(3) $$\sum_{i<\infty} b'_i + b = b.$$

We now put

(4) $$b_n = c_n + \sum_{i<n} a'_i + \sum_{i<\infty} b'_{n+i}.$$

By (1) and (3), this implies in particular

(5) $$b_0 = c_0 + \sum_{i<\infty} b'_i = b + \sum_{i<\infty} b'_i = b.$$

Furthermore, we obtain from (2) and (4) by elementary transformations:

$$b_n = a_n + b_{n+1} \text{ for } n = 0, 1, 2, \cdots .$$

By now applying 1.1.VII to the last formula, and making use of (5), we obtain the conclusion.

Theorem 2.21 provides us with a new method of showing that a given infinite sum is at most equal to a given element, or in particular that two given infinite sums are equal. The method previously applied—e.g., in the proofs of commutative and associative laws 1.38, 1.39, and 1.42—was based on 1.1.VII, or 1.36, and was somewhat awkward; the new method is less elementary, but simpler.

Corollary 2.22. *If $n \cdot a \leqq b$ for every $n < \infty$, then $\infty \cdot a \leqq b$.*
Proof: by 2.21.

Corollary 2.23. *If*

$$\sum_{i<n} a_i = \sum_{i<n} b_i \quad \textit{for every} \quad n < \infty,$$

then

$$\sum_{i<\infty} a_i = \sum_{i<\infty} b_i.$$

PROOF: by 2.21.

THEOREM 2.24. *If* $a_n \leqq a_{n+1}$ *for every* $n < \infty$, *then there is an element* b *(uniquely determined) such that*

(i) $$a_n \leqq b \quad \text{for every} \quad n < \infty,$$

and

(ii) $$a_n \leqq x \quad \text{for every} \quad n < \infty \quad \text{always implies} \quad b \leqq x.$$

PROOF: We construct a sequence $b_0, b_1, \cdots, b_i, \cdots$ with

$$b_0 = a_0, \quad \text{and} \quad a_n + b_{n+1} = a_{n+1} \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

We then have by induction

$$a_n = \sum_{i<n+1} b_i.$$

By putting

$$b = \sum_{i<\infty} b_i,$$

and by applying 2.21, we obtain the conclusion.

In the following theorems, 2.25–2.27 and 2.29, we shall establish certain properties of the set of elements x which satisfy an equation or inequality of the form

$$x + d = e, \quad x + d \leqq e, \quad \text{or} \quad e \leqq x + d$$

where d and e are any two given elements. These theorems and their generalizations 2.38–2.40 will be applied in the next section in the proofs of various distributive laws.

THEOREM 2.25. *If* $0 < n \leqq \infty$, *and* $a_i + d = e$ *for every* $i < n$, *then there are elements* b *and* c *such that* $b + d = c + d = e$, *and* $b \leqq a_i \leqq c$ *for every* $i < n$.

PROOF: The reasoning is analogous to that in the proof of 2.6. We can of course assume that $n = \infty$. We put

(1) $$r_0 = a_0.$$

Assuming that, for a given $m < \infty$, an element r_m with

$$r_m + d = a_{m+1} + d$$

has been defined, we apply 2.6, and obtain three elements r_{m+1}, s_m, and t_m for which

(2) $$r_m = s_m + r_{m+1}, \quad a_{m+1} = t_m + r_{m+1},$$

and

(3) $$d = s_m + d = t_m + d.$$

From (2), (3), and the hypothesis we obtain

$$r_{m+1} + d = a_{m+2} + d;$$

hence the procedure can be continued indefinitely. By 1.47, formula (3) gives

(4) $$\sum_{i<\infty} s_i + d = \sum_{i<\infty} t_i + d = d.$$

By applying 1.1.VII to the first formula of (2), we obtain an element b with

(5) $$r_m = b + \sum_{i<\infty} s_{m+i} \quad \text{for} \quad m = 0, 1, 2, \cdots,$$

whence, by (1),

(6) $$a_0 = b + \sum_{i<\infty} s_i.$$

Finally we put

(7) $$c = a_0 + \sum_{i<\infty} t_i;$$

and we show in an elementary way, by means of (2) and (4)–(7), that the elements b and c satisfy the conclusion.

Corollary 2.26. *If $0 < n \leqq \infty$, and $a_i + d \leqq e$ for every $i < n$, then there is an element c with $c + d = e$, and $a_i \leqq c$ for every $i < n$.*
Proof: by 2.25 (and 1.5).

Theorem 2.27. *If $n \leqq \infty$, and $d \leqq c$, and $e \leqq a_i + d$ for every $i < n$, then there is an element b with $b + d = e$, and $b \leqq a_i$ for every $i < n$.*

Proof: The theorem being obvious in case $n = 0$, we assume that $n \neq 0$. We have for some c

$$c + d = e.$$

Hence, for every $i < n$,

$$c + d \leqq a_i + d;$$

and by means of 2.7 we conclude that there is an element a_i' with

$$a_i' + d = c + d \quad \text{and} \quad a_i' \leqq a_i$$

(a_i' is the element a_1 of the conclusion of 2.7). Since

$$a_i' + d = e \quad \text{for every} \quad i < n,$$

we apply 2.25, and we obtain the element b with the required properties.

THEOREM 2.28 (INTERPOLATION THEOREM). *If $n \leqq \infty$, $p \leqq \infty$, and $a_i \leqq b_j$ for every $i < n$ and $j < p$, then there is an element c such that $a_i \leqq c \leqq b_j$ for every $i < n$ and $j < p$.*

PROOF: We shall distinguish several cases.

(I) $n = p = 2$. We have then, for some d_0, d_1, e_0, and e_1

$$a_0 + d_0 = a_1 + d_1 = b_0, \tag{1}$$

and

$$a_0 + e_0 = a_1 + e_1 = b_1. \tag{2}$$

By applying 2.3 to (1), we obtain

$$a_0 = r_1 + r_2, \quad d_0 = r_3 + r_4, \quad a_1 = r_1 + r_3, \tag{3}$$
$$\text{and} \quad d_1 = r_2 + r_4;$$

hence, with the help of (2),

$$r_1 \leqq a_1 \leqq (e_0 + r_2) + r_1.$$

By 2.27 (with $n = 1$), this inequality implies the existence of an element s for which

$$a_1 = r_1 + s \quad \text{and} \quad s \leqq e_0 + r_2. \tag{4}$$

Hence, by 2.4, we obtain

$$s = s_0 + s_1, \quad s_0 \leqq e_0, \quad \text{and} \quad s_1 \leqq r_2; \tag{5}$$

and from (1)–(5) we infer in an elementary way that the element

$$c = a_0 + s_0$$

satisfies the conclusion.

(II) n arbitrary, $p = 2$; we can obviously assume that $n = \infty$. By (I), we obtain an element c_0 with

$$a_i \leqq c_0 \leqq b_i \quad \text{for} \quad i = 0, 1;$$

then an element c_1 with

$$c_0 \leqq c_1 \quad \text{and} \quad a_2 \leqq c_1 \leqq b_i \quad \text{for} \quad i = 0, 1;$$

and so on, *ad infinitum*. By now applying 2.24 to the increasing sequence $c_0, c_1, \cdots, c_i, \cdots$, we get an element c which satisfies the conclusion.

(III) $n = 2$, p arbitrary; we can assume that $p \neq 0$. By (I) we obtain, for any given $j < p$, an element c_j with

(6) $$a_i \leqq c_j \leqq a_0 + a_1 \quad \text{for} \quad i = 0, 1, \quad \text{and} \quad c_j \leqq b_j.$$

We have

$$c_j \leqq a_0 + a_1 \leqq a_i + c_j;$$

and hence, by 2.27 (with $n = 2$), we get an element d_j with

(7) $$d_j + c_j = a_0 + a_1 \quad \text{and} \quad d_j \leqq a_i \quad \text{for} \quad i = 0, 1.$$

By (II), the latter inequality implies the existence of an element e such that

(8) $$d_j \leqq e \leqq a_i \quad \text{for} \quad j < p \quad \text{and} \quad i = 0, 1.$$

By (6) and (8) we can put

(9) $$a_0 + r_j = c_j \quad \text{and} \quad e + s = a_1.$$

We easily derive from (7)–(9)

$$a_0 + e \leqq s + (a_0 + e) \leqq r_j + (a_0 + e) \quad \text{for} \quad j < p.$$

Hence, by applying 2.27 again, we get

(10) $$t + (a_0 + e) = s + (a_0 + e), \quad \text{and} \quad t \leqq r_j \quad \text{for} \quad j < p$$

We put

(11) $$c = a_0 + t.$$

We have by (8)–(11)

$$a_1 + e \leqq c + e \quad \text{and} \quad e \leqq c,$$

whence, by 2.13,

$$a_1 \leqq c.$$

From (6) and (9)–(11) it is also easily seen that

$$a_0 \leqq c \leqq b_j \quad \text{for} \quad j < p.$$

Thus, c has the required properties.

(IV) n and p arbitrary. We reason as in (II), using (III) instead of (I).

THEOREM 2.29. *If $n \leqq \infty$, and $e \leqq a_i + d$ for every $i < n$, then there is an element b such that $e \leqq b + d$, and $b \leqq a_i$ for every $i < n$.*

PROOF: By 2.28 (with $n = 2$ and $p = n$) there is an element c such that

$$e \leqq c, \quad \text{and} \quad d \leqq c \leqq a_i + d \quad \text{for} \quad i < n.$$

Hence, if $n \neq 0$, we obtain the conclusion by 2.27; if $n = 0$, we put $b = e$.

THEOREM 2.30 (MEAN VALUE THEOREM). *If $0 < n \leqq \infty$, $0 < p \leqq \infty$, $a_i \leqq b_j$ and $a_i + d \leqq e \leqq b_j + d$ for every $i < n$ and $j < p$, then there is an element c such that $e = c + d$, and $a_i \leqq c \leqq b_j$ for every $i < n$ and $j < p$.*

PROOF: By 2.26 there is an element c' such that

(1) $$c' + d = e, \quad \text{and} \quad a_i \leqq c' \quad \text{for} \quad i < n.$$

We put

(2) $$b'_0 = c', \quad \text{and} \quad b'_{j+1} = b_j \quad \text{for} \quad j < p.$$

Then, by (1) and the hypothesis,

$$d \leqq e \leqq b'_j + d \quad \text{for} \quad j < p + 1;$$

and hence by 2.27 we get an element c'' with

(3) $$c'' + d = e, \quad \text{and} \quad c'' \leqq b'_j \quad \text{for} \quad j < p + 1.$$

By putting

(4) $$a'_0 = c'', \quad \text{and} \quad a'_{i+1} = a_i \quad \text{for} \quad i < n,$$

we have by (3) and the hypothesis

$$a'_i \leqq b'_j \quad \text{for} \quad i < n + 1 \quad \text{and} \quad j < p + 1;$$

and hence by 2.28

(5) $$a'_i \leqq c \leqq b'_j \text{ for } i < n+1 \text{ and } j < p+1.$$

By means of (1)–(5) we easily show that c satisfies the conclusion.

From 2.30 we can derive 2.25–2.29 in an elementary way. When applied to the case $n = p = 1$, Theorem 2.30 shows that the function

$$f(x) = x + d$$

satisfies a familiar theorem on continuous functions in the real domain; i.e., considered in the 'interval' with the endpoints a and b $(a \leqq b)$, it assumes every intermediate value between the two extremes $f(a)$ and $f(b)$.

The proof of the following theorem which implies as immediate corollaries the important CANCELLATION LAWS FOR MULTIPLES is perhaps the most involved in the whole arithmetic of C.A.'s.

THEOREM 2.31. *If* $0 < m < \infty$ *and* $m \cdot a + c \leqq m \cdot b + c$, *then* $a + c \leqq b + c$.

PROOF: We begin with a lemma:

Let T *be a set of triples such that, for every* $\langle a, b, c\rangle \,\varepsilon\, T$, *there is another triple* $\langle a', b', c'\rangle \,\varepsilon\, T$ *with*

$$a + c = 2 \cdot a' + c' \quad \textit{and} \quad b + c = a' + b' + c'.$$

We then have $a + c \leqq b + c$ *for every triple* $\langle a, b, c\rangle \,\varepsilon\, T$.

In fact, for any given $\langle a, b, c\rangle \,\varepsilon\, T$ we construct by recursion three infinite sequences:

(1) $$a_0 = a, \quad b_0 = b, \quad c_0 = c,$$

(2) $$a_n + c_n = a_{n+1} + (a_{n+1} + c_{n+1}) \quad \text{and}$$

$$b_n + c_n = a_{n+1} + b_{n+1} + c_{n+1} \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

From the first formula of (2) we conclude by 1.1.VII (with 'a_n' and 'b_n' changed to '$a_n + c_n$' and 'a_{n+1}') that

(3) $$a_n + c_n = d + \sum_{i<\infty} a_{n+i+1}$$

for some d and for every $n < \infty$; in particular, by (1),

(4) $$a + c = d + \sum_{i<\infty} a_{i+1}.$$

From (1) and (2) we obtain by induction

$$b + c = b_n + c_n + \sum_{i<n} a_{i+1};$$

and since (2) and (3) imply

$$d \leqq b_n + c_n ,$$

we arrive at

(5) $$d + \sum_{i<n} a_{i+1} \leqq b + c \quad \text{for every} \quad n < \infty .$$

The conclusion of our lemma follows from (4) and (5) by 2.21.

Consider now any three elements a, b, and c for which

$$2\cdot a + c \leqq 2\cdot b + c.$$

We put

$$2\cdot a + c + d = 2\cdot b + c,$$

whence

$$(a + c) + (a + c + d) = 2\cdot(b + c).$$

By applying 2.5 to the latter formula, we obtain elements a', b', and c' such that

$$a + c = 2\cdot a' + c', \quad a + c + d = 2\cdot b' + c',$$
$$\text{and} \quad b + c = a' + b' + c'.$$

Thus the set T of all triples $\langle a, b, c\rangle$ with

$$2\cdot a + c \leqq 2\cdot b + c$$

satisfies the hypothesis of our lemma; and hence we conclude by means of this lemma that 2.31 holds for $m = 2$. We extend this result by induction to all numbers m of the form $m = 2^l$.

Now let

(6) $$m\cdot a + c \leqq m\cdot b + c$$

where m is an arbitrary number with $1 < m < \infty$. We again put

(7) $$m\cdot a + c + d = m\cdot b + c.$$

If l is a number with

$$2^l \leqq m < 2^{l+1},$$

we have by (6)

$$2^l\cdot a + c \leqq 2^l\cdot(2\cdot b) + c;$$

hence, by the result previously established,

$$a + c \leqq 2 \cdot b + c,$$

and we put

$$(8) \qquad a + c + e = 2 \cdot b + c.$$

From (7) and (8) we easily derive

$$(m - 1) \cdot (c + e) + (m - 1) \cdot a$$
$$= [(m - 2) \cdot b + (m - 1) \cdot c + d] + m \cdot a;$$

by applying 2.10 (with $n = m - 1$ and $c = a$), we obtain

$$(m - 1) \cdot (c + e) \leqq (m - 2) \cdot b + (m - 1) \cdot c + d + a,$$

and consequently we put

$$(9) \qquad (m - 1) \cdot (c + e) + f = (m - 2) \cdot b + (m - 1) \cdot c + d + a.$$

A series of elementary transformations leads from (7)–(9) to

$$(2 \cdot m - 2) \cdot b + (m - 1) \cdot c + f = (2 \cdot m - 2) \cdot b + (m - 1) \cdot c;$$

and by applying 2.12 twice, we get

$$(10) \qquad b + c + f = b + c.$$

Hence by (8)

$$(a + c) + (c + e + f) = 2 \cdot (b + c).$$

We apply 2.5 to the latter formula, and we obtain elements a', b', and c' with

$$(11) \qquad a + c = 2 \cdot a' + c', \quad c + e + f = 2 \cdot b' + c', \quad \text{and}$$
$$b + c = a' + b' + c'.$$

(9) and (10) imply

$$a + c + (m - 2) \cdot (b + c) = a + c + (m - 2) \cdot (b + c + f)$$
$$\leqq (m - 1) \cdot (c + e + f),$$

whence by (11)

$$m \cdot a' + c' + (m - 2) \cdot (b' + c')$$
$$\leqq (m - 1) \cdot b' + (m - 1) \cdot (b' + c');$$

and, by means of 2.10 (with $n = m - 2$ and $c = b' + c'$), we arrive at

(12) $$m \cdot a' + c' \leqq m \cdot b' + c'.$$

Thus, (6) implies the existence of elements a', b', and c' which satisfy (11) and (12); this shows that our lemma can be applied to the set T of all triples $\langle a, b, c \rangle$ with

$$m \cdot a + c \leqq m \cdot b + c.$$

Consequently, 2.31 holds for every m, $0 < m < \infty$.

COROLLARY 2.32. *If $0 < m < \infty$ and $m \cdot a + c = m \cdot b + c$, then $a + c = b + c$.*

PROOF: by 2.31.

COROLLARY 2.33. *If $0 < m < \infty$ and $m \cdot a \leqq m \cdot b$, then $a \leqq b$.*

PROOF: by 2.31.

COROLLARY 2.34.[8] *If $0 < m < \infty$ and $m \cdot a = m \cdot b$, then $a = b$.*

PROOF: by 2.32 or 2.33.

THEOREM 2.35. *If $0 < m < \infty$ and $n < \infty$, then $m \cdot a + n \cdot c \leqq m \cdot b + n \cdot c$ implies $a + c \leqq b + c$; and if $m \leqq \infty$ and $0 < n \leqq \infty$, then the converse implication holds.*

PROOF: first part by 2.11 and 2.31; second part by 2.17.

COROLLARY 2.36. *If $0 < m < \infty$ and $n < \infty$, then $m \cdot a + n \cdot c = m \cdot b + n \cdot c$ implies $a + c = b + c$; and if $m \leqq \infty$ and $0 < n \leqq \infty$, then the converse implication holds.*

PROOF: by 2.35.

THEOREM 2.37 (EUCLID'S THEOREM).[9] *If the numbers $m < \infty$ and $n < \infty$ are relatively prime and if $m \cdot a = n \cdot b$, then there is an element c such that $a = n \cdot c$ and $b = m \cdot c$.*

[8] Corollary 2.34 for cardinal numbers is a result of F. Bernstein; the proof for $m = 2$ and an outline of the proof for the general case can be found in Bernstein [1], pp. 122 ff. Another, simpler proof for $m = 2$ was given in Sierpiński [2]; and for an arbitrary m was found (but not published) by A. Lindenbaum. Corollary 2.33, which is a generalization of 2.34, was proved in the case of $m = 2$ by the author, and in the general case by A. Lindenbaum. The author has been unable to carry over the earlier proofs of 2.33 and 2.34 to the abstract theory of cardinal algebras; the proof of 2.31 outlined above seems to have but very little in common with those proofs.

[9] For cardinal numbers this theorem was obtained by A. Lindenbaum.

PROOF: We can assume that $m \neq 0$. As is well known, the hypothesis implies that there are numbers $k < \infty$ and $l < \infty$ for which

(1) $$k \cdot m = l \cdot n + 1.$$

We easily obtain by (1) and the hypothesis

$$m \cdot (l \cdot a) \leqq m \cdot (k \cdot b);$$

hence, by 2.33,

$$l \cdot a \leqq k \cdot b,$$

and we can put

(2) $$l \cdot a + d = k \cdot b.$$

From (1) and (2), by a few elementary steps, we arrive at

$$m \cdot d + (l \cdot n) \cdot b \leqq (l \cdot n + 1) \cdot b;$$

and, by applying 2.10 (with $b = 0$ and $c = b$), we get

$$m \cdot d \leqq b.$$

Consequently, we put

(3) $$m \cdot d + e = b.$$

We easily obtain, by (1)–(3) and the hypothesis,

$$(k \cdot m) \cdot b = e + (k \cdot m) \cdot b.$$

An application of 2.12 (with $a = 0$, $b = e$, and $c = b$) gives

$$b = e + b,$$

and further, e.g. by 1.46 and (3),

(4) $$b = b + (m - 1) \cdot e = m \cdot (d + e).$$

Hence, from the hypothesis,

$$m \cdot a = m \cdot [n \cdot (d + e)],$$

and therefore by 2.34

(5) $$a = n \cdot (d + e).$$

Thus, by (4) and (5), the element

$$c = d + e$$

satisfies the conclusion.

With the help of the cancellation laws for multiples we can now generalize the results obtained in 2.25, 2.26, and 2.29; this will be done in 2.38–2.40.

THEOREM 2.38.[10] *If $m \leqq \infty$, $n \leqq \infty$, but neither $m = n = 0$ nor $m = n = \infty$, and if $e \leqq m \cdot a_i + d$ for every $i < n$, then there is an element b such that $e \leqq m \cdot b + d$, and $b \leqq a_i$ for every $i < n$.*

PROOF: (I) $m < \infty$. We proceed by induction with respect to m. If $m = 0$ (and hence $n \neq 0$), we take $b = 0$. We assume now that the theorem holds for $m = k$; and that

$$e \leqq (k + 1) \cdot a_i + d \quad \text{for} \quad i < n.$$

Given any number $j < n$, we obtain

$$(k + 1) \cdot e \leqq (k + 1) \cdot (k \cdot a_i + a_j + d)$$

and further, by 2.33,

$$e \leqq k \cdot a_i + (a_j + d).$$

Hence, by our inductive premise, there is an element b_j with

(1) $$e \leqq k \cdot b_j + (a_j + d) \quad \text{and} \quad b_j \leqq a_i \quad \text{for} \quad i < n \quad \text{and} \quad j < n.$$

By applying 2.28 to the second part of (1), we get an element c with

(2) $$b_j \leqq c \leqq a_i \quad \text{for} \quad i < n \quad \text{and} \quad j < n.$$

Hence by (1)

$$e \leqq a_j + (k \cdot c + d);$$

therefore, by 2.29, we have for some b'

(3) $$e \leqq b' + (k \cdot c + d), \quad \text{and} \quad b' \leqq a_j \quad \text{for} \quad j < n.$$

[10] The proof of Theorem 2.38 (which implies the distributive law 3.27 in §3) is a joint result of B. Jónsson and the author.

We now have by (2) and (3)

$$b' \leqq a_i \text{ and } c \leqq a_i \text{ for } i < n;$$

and, by applying 2.28 (with $n = 2$ and $p = n$), we obtain an element b with

(4) $$b' \leqq b, \quad c \leqq b, \quad \text{and} \quad b \leqq a_i \quad \text{for} \quad i < n.$$

From (3) and (4) it is easily seen that b satisfies the conclusion for $m = k + 1$.

(II) $m = \infty$. Hence, by hypothesis, $n < \infty$, and we proceed by induction with respect to n. If $n = 0$, we take $b = c$. We assume now that the theorem holds for $n = l$, and that

$$e \leqq \infty \cdot a_i + d \quad \text{for} \quad i < l + 1.$$

By the inductive premise, there is an element b' with

(5) $$e \leqq \infty \cdot b' + d, \quad \text{and} \quad b' \leqq a_i \quad \text{for} \quad i < l;$$

since, moreover,

$$e \leqq \infty \cdot a_l + d,$$

we obtain by 2.29 ($n = 2$) an element b'' such that

(6) $$e \leqq b'' + d \quad \text{with} \quad b'' \leqq \infty \cdot b' \quad \text{and} \quad b'' \leqq \infty \cdot a_l .$$

By applying 2.2 to the last two inequalities, we arrive at two refinements of b'':

$$b'' = \sum_{i<\infty} r_i = \sum_{j<\infty} s_j \quad \text{with} \quad r_i \leqq b' \quad \text{for} \quad i < \infty, \quad \text{and}$$

$$s_j \leqq a_l \quad \text{for} \quad j < \infty.$$

Hence, by 2.1, we have

(7) $$b'' = \sum_{i<\infty} \sum_{j<\infty} t_{i,j} \quad \text{with} \quad t_{i,j} \leqq b' \quad \text{and} \quad t_{i,j} \leqq a_l .$$

We can transform the double sequence $t_{i,j}$ into a simple one and apply 2.28 (with $n = \infty, p = 2$) to the last two inequalities of (7); we obtain an element b with

(8) $$t_{i,j} \leqq b \quad \text{for} \quad i < \infty \quad \text{and} \quad j < \infty, \quad b \leqq b', \quad \text{and} \quad b \leqq a_l .$$

(6)–(8) easily imply

$$e \leqq \infty \cdot b + d,$$

while (5) and (8) give

$$b \leqq a_i \quad \text{for} \quad i < l + 1.$$

Thus, b satisfies the conclusion for $n = l + 1$, and the proof is complete.

By means of a simple example from the C.A. of non-negative real numbers (discussed below in §14), it can be shown that 2.38 may fail in case $m = n = \infty$.

THEOREM 2.39. *If $m \leqq \infty$, $0 < n \leqq \infty$, and $m \cdot a_i + d \leqq e$ for every $i < n$, then there is an element c such that $m \cdot c + d \leqq e$, and $a_i \leqq c$ for every $i < n$.*

PROOF: If $m < \infty$, we reason as in the proof of 2.38, with 2.29 replaced by 2.26. If $m = \infty$, we obtain by 1.29 and 1.46

$$a_i + e = e \quad \text{and} \quad \infty \cdot a_i + e = e \quad \text{for} \quad i < n.$$

Hence, with the help of 1.47,

$$\infty \cdot \sum_{i<n} a_i + e = \sum_{i<n} (\infty \cdot a_i) + e = e;$$

and since, by hypothesis,

$$d \leqq e,$$

we conclude that

$$\infty \cdot \sum_{i<n} a_i + d \leqq e.$$

Thus, the element

$$c = \sum_{i<n} a_i$$

satisfies the conclusion.

COROLLARY 2.40. *If $m \leqq \infty$, $0 < n \leqq \infty$, but not $m = n = \infty$, and if $m \cdot a_i + d = e$ for every $i < n$, then there are elements b and c such that $m \cdot b + d = m \cdot c + d = e$, and $b \leqq a_i \leqq c$ for every $i < n$. The element c (but not necessarily b) exists also in case $m = n = \infty$.*

PROOF: by 2.38 and 2.39.

Corollary 2.40 shows that, in the set of all solutions x of an equation

$$m \cdot x + d = e$$

(m an arbitrary number $< \infty$, d and e arbitrary elements), every sequence has a lower bound and an upper bound which belong to this set. By applying successively 2.40 and 2.28, we can show that this still holds for the set of common solutions x of every finite or infinite sequence of such equations:

$$m_j \cdot x + d_j = e_j, \quad j = 0, 1, 2, \cdots.$$

The results obtained in this section permit us, among other things, to develop the theory of REAL MULTIPLES $r \cdot a$ where r is an arbitrary non-negative real number and a an arbitrary element of our algebra. We first define this notion in an obvious way for rational numbers, and then extend it (e.g., by using sums of infinite sequences) to irrational numbers. In the rational case the unicity of $r \cdot a$ follows immediately from 2.34; in the irrational case the proof is more involved. It is not true, of course, that the multiple $r \cdot a$ exists for any given r and a (though this may be the case in some particular C.A.'s). Regarding this problem the following results can be established (with the help of 2.34 and 2.37):

I. *If, for a given element a, there is a largest integer $n < \infty$ such that $a = n \cdot b$ for some b, then, r being a non-negative real number, $r \cdot a$ exists in case $r \cdot n$ is an integer, and does not exist otherwise.*

II. *If, for a given element a, there is no largest integer $n < \infty$ such that $a = n \cdot b$ for some b, then $r \cdot a$ exists for every non-negative real number r.*

As further important theorems on real multiples we mention the two distributive laws:

III. *If $n \leqq \infty$, and $r_0, r_1, \cdots, r_i, \cdots$ with $i < n$ are non-negative real numbers, and the elements $r_i \cdot a$ exist for every $i < n$, then $(\sum_{i<n} r_i) \cdot a$ also exists, and*

$$(\sum_{i<n} r_i) \cdot a = \sum_{i<n} (r_i \cdot a).$$

IV. *If $n \leqq \infty$, r is a non-negative real number, and the elements $r \cdot a_i$ exist for every $i < n$, then $r \cdot \sum_{i<n} a_i$ also exists, and*

$$r \cdot \sum_{i<n} a_i = \sum_{i<n} (r \cdot a_i).$$

In connection with III it should be pointed out that we regard ∞ as a real number, and we put

$$\sum_{i<n} r_i = \infty$$

in case $n = \infty$ and the sum is divergent in the ordinary sense, or in case at least one of the numbers r_i equals ∞.

§ 3. GREATEST LOWER BOUND AND LEAST UPPER BOUND

As we know from §1, the relation $\leqq$ establishes in every C.A. a partial order. Hence we can apply to C.A.'s various notions of the general theory of partially ordered sets; and in the first place, those of the GREATEST LOWER BOUND and the LEAST UPPER BOUND.[11]

DEFINITION 3.1. *Let I be an arbitrary set; and let an element $a_i \varepsilon A$ be correlated with every element $i \varepsilon I$.*

(i) *By the GREATEST LOWER BOUND OF THE ELEMENTS a_i we understand the unique element $a \varepsilon A$ which satisfies the conditions: $a \leqq a_i$ for every $i \varepsilon I$; and, if $x \leqq a_i$ for every $i \varepsilon I$, then $x \leqq a$. In symbols,*

$$a = \bigcap_{i \varepsilon I} a_i .$$

(ii) *If I consists of two numbers, 0 and 1, and if $a_0 = b$ and $a_1 = c$, we put*

$$a = b \cap c.$$

DEFINITION 3.2. *Let 'I' and 'a_i' have the same meaning as in 3.1.*

(i) *By interchanging in 3.1(i) the sides of all inequalities (or by changing '$\leqq$' to '$\geqq$'), we arrive at the notion of the LEAST UPPER BOUND a OF THE ELEMENTS a_i. In symbols,*

$$a = \bigcup_{i \varepsilon I} a_i .$$

(ii) *In the particular case considered in 3.1(ii), we use the notation*

$$a = b \cup c.$$

We are going to state here the most important properties of the notions just defined. However, we shall be interested in this work almost exclusively in bounds of two elements and of finite and infinite sequences of elements, i.e., in the case when I consists of all integers i with $i < n \leqq \infty$. We omit various elementary consequences of 3.1 and 3.2 which hold in arbitrary partially ordered sets and are not specific for C.A.'s; such consequences are assumed to be known.[12] For an analogous reason, when applying Definitions 3.1

[11] Consult in this connection Birkhoff [3].
[12] See again Birkhoff [3].

and 3.2, we shall not always refer to them explicitly. Since the bounds of a sequence of elements, or even of two elements, may not exist, most theorems will be provided with assumptions which secure the existence of all or some of the bounds involved. Such assumptions, if required, will always be explicitly stated in case they concern the existence of a bound which is involved in the conclusion of a theorem, without being involved in its hypothesis.

LEMMA 3.3. *If* $a \leqq b$, $a \leqq c + d$, *and* $b \cap c$ *exists, then*

$$a \leqq (b \cap c) + d.$$

PROOF: By 2.4 we have

$$a = a_1 + a_2 \quad \text{with} \quad a_1 \leqq c \quad \text{and} \quad a_2 \leqq d.$$

Hence and by 3.1

$$a \leqq a_1 + d \quad \text{and} \quad a_1 \leqq b \cap c.$$

The conclusion follows at once.

THEOREM 3.4. *If* $a \cap b$ *exists, then* $a \cup b$ *also exists, and* $(a \cap b) + (a \cup b) = a + b$.

PROOF: We have

$$a \cap b \leqq a \quad \text{and} \quad a \cap b \leqq b.$$

Hence, for some c,

$$(a \cap b) + c = a. \tag{1}$$

Obviously

$$a \leqq b + c \quad \text{and} \quad b \leqq b + c.$$

If now

$$a \leqq x \quad \text{and} \quad b \leqq x,$$

we have

$$b \leqq a + d = x,$$

and consequently, by 3.3 (with $a = b$ and $c = a$) and (1),

$$b + c \leqq (a \cap b) + d + c = a + d = x.$$

Thus, by 3.2,

(2) $$b + c = a \cup b.$$

The conclusion is implied by (1) and (2).

In connection with 3.4 it should be mentioned that the existence of $a \cup b$ does not necessarily imply the existence of $a \cap b$.

COROLLARY 3.5. *If $a \cap b = 0$, then $a \cup b = a + b$.*
PROOF: by 3.4.

Two elements a and b which satisfy the hypothesis of 3.5 are said to be DISJOINT (or MUTUALLY EXCLUSIVE). The converse of 3.5 does not hold; we have, for instance, for every element a

$$a \cup \infty \cdot a = a + \infty \cdot a = \infty \cdot a \quad \text{and} \quad a \cap \infty \cdot a = a.$$

It may be mentioned in this connection that, by 3.1, the hypothesis of 3.5 amounts to saying that, for every x, $x \leqq a$ and $x \leqq b$ imply $x = 0$; while, as regards the formula in the conclusion of 3.5, we have

THEOREM 3.6. *In order that*

(i) $$a \cup b = a + b,$$

it is necessary and sufficient that
(ii) *for every x, the formulas $x \leqq a$ and $x \leqq b$ imply*

$$x + a + b = a + b.$$

PROOF: Assume (i) to hold. If

$$x \leqq a \quad \text{and} \quad x \leqq b,$$

we have for some y

$$x + y = b, \quad b \leqq a + y, \quad a \leqq a + y,$$

and hence by (i)

$$a + b \leqq a + y \quad \text{and} \quad x + a + b \leqq x + a + y = a + b.$$

Thus, (ii) is satisfied. If, conversely, (ii) holds, consider an element z with

(1) $$a \leqq z \quad \text{and} \quad b \leqq z.$$

Let

$$a + a' = b + b' = z.$$

Then 2.3 gives

$$a = x_1 + x_2, \quad a' = x_3 + x_4, \quad b = x_1 + x_3, \quad b' = x_2 + x_4.$$

By (ii) we obtain

$$x_1 + a + b = a + b, \quad \text{i.e.,} \quad x_1 + x_2 + x_3 + 2 \cdot x_1 = x_2 + x_3 + 2 \cdot x_1.$$

Now by 2.12

$$x_2 + x_3 + 2 \cdot x_1 = x_2 + x_3 + x_1;$$

and hence, easily,

$$a + b \leqq z. \tag{2}$$

Thus, (1) implies (2) for every z. From this we conclude that (i) holds, and the proof is complete.

THEOREM 3.7. *If $a \cap c \leqq b$ and $a \leqq b \cup c$, then $a \leqq b$.*

PROOF: We have for some d

$$(a \cap c) + d = b.$$

Hence successively

$$b \leqq c + d, \quad b \cup c \leqq c + d, \quad \text{and} \quad a \leqq c + d.$$

By applying 3.3 (with $b = a$) to the last formula, we arrive at the conclusion.

COROLLARY 3.8. *If $a \cap c = b \cap c$ and $a \cup c = b \cup c$, then $a = b$.*

PROOF: by 3.7.

THEOREM 3.9. *If $a \cap c = b \cap c$ and $a + c = b + c$, then $a = b$.*

PROOF: We have by 3.4 and the hypothesis

$$(a \cup c) + (a \cap c) = (b \cup c) + (a \cap c), \; a \cap c \leqq a \cup c, \quad \text{and}$$

$$a \cap c \leqq b \cup c.$$

Hence, by 2.15,

$$a \cup c = b \cup c.$$

The conclusion follows by 3.8.

THEOREM 3.10. *If $n \leqq \infty$, and the elements $a \cap b_i$ exist for every $i < n$, then*

(i) $$c \leqq a \quad \text{and } c \leqq \sum_{i<n} b_i \quad \text{imply} \quad c \leqq \sum_{i<n} (a \cap b_i);$$

(ii) *if, in addition,* $a \cap \sum_{i<n} b_i$ *exists, then*

$$a \cap \sum_{i<n} b_i \leqq \sum_{i<n} (a \cap b_i).$$

PROOF: analogous to that of 3.3, with 2.4 replaced by 2.2.

An inequality analogous to 3.10(ii), with '$\cap$' replaced by '$\cup$', also holds, but is quite trivial.

THEOREM 3.11. *If $m < n \leqq \infty$, $a \cap b_i = 0$ for every i with $i < n$ and $i \neq m$, and either $a \cap b_m$ or $a \cap \sum_{i<n} b_i$ exists, then*

$$a \cap \sum_{i<n} b_i = \sum_{i<n} (a \cap b_i) = a \cap b_m .$$

PROOF: Assume that $a \cap b_m$ exists. We have

$$a \cap b_m \leqq a \quad \text{and} \quad a \cap b_m \leqq b_m \leqq \sum_{i<n} b_i ;$$

and by 3.10

$$x \leqq a \quad \text{and} \quad x \leqq \sum_{i<n} b_i \quad \text{imply} \quad x \leqq \sum_{i<n} (a \cap b_i) = a \cap b_m .$$

Hence the conclusion. Similarly under the alternative assumption.

COROLLARY 3.12. *If $n \leqq \infty$, and $a \cap b_i = 0$ for every $i < n$, then*

$$a \cap \sum_{i<n} b_i = 0.$$

PROOF: by 3.11.

THEOREM 3.13. *If $n \leqq \infty$, $b_i \cap b_j = 0$ for all i and j with $i < j < n$, and*

$$a \leqq \sum_{i<n} b_i ,$$

then the elements $a \cap b_i$ for $i < n$ exist, and they are the only elements a_i which satisfy the conditions:

$$a = \sum_{i<n} a_i , \quad \text{and} \quad a_i \leqq b_i \quad \text{for every} \quad i < n.$$

PROOF: By 2.2, there exists at least one sequence of elements which satisfy the conditions of the theorem. If now $a_0, a_1, \cdots, a_i, \cdots$ is any such sequence, we have, for any given $m < n$,

$$a_m \leqq a \quad \text{and} \quad a_m \leqq b_m .$$

Furthermore, if

$$x \leqq a \quad \text{and} \quad x \leqq b_m ,$$

we clearly have

$$x \cap a_i = 0 \quad \text{for} \quad i \neq m, \quad i < n, \quad \text{and} \quad x \cap \sum_{i<n} a_i = x.$$

Hence, by 3.11,

$$x \cap a_m = x \leqq a_m .$$

Consequently

$$a_m = a \cap b_m \quad \text{for every} \quad m < n.$$

This completes the proof.

THEOREM 3.14. *If $n \leqq \infty$, $p \leqq \infty$, $b_j \cap b_k = 0$ for all j and k with $j < k < p$, and*

$$\sum_{i<n} a_i \leqq \sum_{j<p} b_j ,$$

then all the elements $a_i \cap b_j$ with $i < n$ and $j < p$ exist, and we have

$$(\sum_{i<n} a_i) \cap b_j = \sum_{i<n} (a_i \cap b_j) \quad \textit{for every} \quad j < p.$$

PROOF: By 3.13, the elements $a_i \cap b_j$ exist, and we have

$$a_i = \sum_{j<p} (a_i \cap b_j) \quad \text{for} \quad i < n,$$

whence

$$\sum_{i<n} a_i = \sum_{j<p} [\sum_{i<n} (a_i \cap b_j)]. \tag{1}$$

For any given $j < p$ we have by 3.12 and the hypothesis

$$[\sum_{i<n} (a_i \cap b_j)] \cap b_k = 0 \quad \text{for} \quad k < p, \quad k \neq j,$$

as well as

$$\sum_{i<n} (a_i \cap b_j) \cap \sum_{k<p} b_k = \sum_{i<n} (a_i \cap b_j).$$

Hence, by applying 3.11 with appropriate substitutions,

$$(2) \qquad \sum_{i<n} (a_i \cap b_j) = [\sum_{i<n} (a_i \cap b_j)] \cap b_j \leqq b_j \quad \text{for} \quad j < p.$$

By 3.13, formulas (1) and (2) imply the conclusion.

THEOREM 3.15. *If* $n \leqq \infty$, $a + \sum_{i<n} b_i = \sum_{i<n} b_i$, *and* $b_i \cap b_j = 0$ *for all* i *and* j *with* $i < j < n$, *then the elements* $a \cap b_i$ *exist, and we have*

$$a = \sum_{i<n} (a \cap b_i), \quad \textit{and} \quad (a \cap b_i) + b_i = b_i \quad \textit{for every} \quad i < n.$$

PROOF: The existence of $a \cap b_i$ and the first formula of the conclusion result directly from 3.13. To obtain the second formula, we apply 3.14 with

$$n = 2, \quad p = n, \quad a_0 = a, \quad \text{and} \quad a_1 = \sum_{i<n} b_i.$$

Further theorems on bounds of two elements can be obtained from more general results concerning bounds of sequences, to which we now turn.

THEOREM 3.16. *Let* n *be* $\leqq \infty$. *In order that*

$$a = \bigcap_{i<n} a_i$$

it is necessary and sufficient that

$$\text{(i)} \qquad a \leqq a_i \quad \textit{for every} \quad i < n,$$

and that

$$\text{(ii)} \qquad \textit{if } a \leqq x \leqq a_i \quad \textit{for every} \quad i < n, \textit{ then } x = a.$$

PROOF: The necessity of conditions (i) and (ii) follows directly from 3.1. Now assume (i) and (ii) to hold. If

$$x \leqq a_i \quad \text{for} \quad i < n,$$

we apply 2.28 (with $n = 2$ and $p = n$), and we obtain an element c for which

$$x \leqq c, \quad a \leqq c, \quad \text{and} \quad c \leqq a_i \quad \text{for} \quad i < n.$$

Hence, by (ii),

$$c = a \quad \text{and} \quad x \leqq a.$$

Consequently, by 3.1 and (i),

$$a = \bigcap_{i<n} a_i .$$

This completes the proof.

THEOREM 3.17. *Let n be $\leqq \infty$. In order that*

$$a = \bigcup_{i<n} a_i$$

it is necessary and sufficient that

(i) $\quad a_i \leqq a$ *for every* $i < n$,

and that

(ii) *if* $a_i \leqq x \leqq a$ *for every* $i < n$, *then* $x = a$.

PROOF: entirely analogous to that of 3.16, with 3.1 replaced by 3.2.

THEOREM 3.18. *If $n \leqq \infty$ and $\bigcup_{i<n} a_i$ exists, then*

$$\bigcup_{i<n} a_i \leqq \sum_{i<n} a_i .$$

PROOF: by 3.2.

THEOREM 3.19. $\sum_{i<\infty} a_i = \bigcup_{n<\infty} \sum_{i<n} a_i$.
PROOF: by 2.21.

Theorem 3.19 provides us with a new and simpler characterization of infinite addition in terms of binary addition (cf. the remarks which follow 1.17).

COROLLARY 3.20. $\infty \cdot a = \bigcup_{n<\infty} (n \cdot a)$.
PROOF: by 3.19.

THEOREM 3.21. *If $a_n \leqq a_{n+1}$ for every $n < \infty$, then $\bigcup_{i<\infty} a_i$ exists.*
PROOF: by 2.24.

COROLLARY 3.22. *If $\bigcup_{i<n} a_i$ exists for every $n < \infty$, then $\bigcup_{i<\infty} a_i$ also exists, and we have*

$$\bigcup_{i<\infty} a_i = \bigcup_{n<\infty} \bigcup_{i<n} a_i .$$

PROOF: by 3.21.

A generalization of 3.5 is

THEOREM 3.23. *If* $n \leqq \infty$, *and* $a_i \cap a_j = 0$ *for all* i *and* j *with* $i < j < n$, *then*

$$\bigcup_{i<n} a_i = \sum_{i<n} a_i .$$

PROOF: In case $n < \infty$ we proceed by induction based upon 3.5 and 3.12. To extend the result to the case $n = \infty$, we apply 3.19 and 3.22.

In connection with 3.21 and 3.22 it should be mentioned that analogous theorems regarding decreasing sequences and lower bounds in general fail; this will be seen from Theorem 15.12 below. We have, however,

THEOREM 3.24. *If* $a_n = b_n + a_{n+1}$ *and* $b_n \cap a_{n+1} = 0$ *for every* $n < \infty$, *then the element* $\bigcap_{i<\infty} a_i$ *exists; and, in fact, it is the only element* c *such that*

$$a_n = c + \sum_{i<\infty} b_{n+i} \quad \textit{for every} \quad n < \infty .$$

PROOF: The existence of an element c with the indicated property follows from 1.1.VII. Now let c be any such element. We have

$$c \leqq a_n \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

If

$$x \leqq a_n \quad \text{for} \quad n = 0, 1, 2, \cdots ,$$

we obtain by the hypothesis and 3.12

$$x \leqq \sum_{i<\infty} b_i + c = a_0 \quad \text{and} \quad x \cap \sum_{i<\infty} b_i = 0,$$

and hence by 3.11

$$x \leqq c.$$

Thus,

$$c = \bigcap_{i<\infty} a_i ,$$

which completes the proof.

A sequence $a_0, a_1, \cdots, a_i, \cdots$ for which there is another sequence $b_0, b_1, \cdots, b_i, \cdots$ with $a_n = b_n + a_{n+1}$ and $b_n \cap a_{n+1} = 0$ for every

$n < \infty$ may be called DISJOINTLY DECREASING. Thus, by 3.24, every disjointly decreasing sequence has a greatest lower bound.

A series of important distributive laws 3.25–3.32 follows.

THEOREM 3.25. *If* $n \leqq \infty$ *and* $\bigcap_{i<n} b_i$ *exists, then*

$$a + \bigcap_{i<n} b_i = \bigcap_{i<n} (a + b_i).$$

PROOF: We have

$$a + \bigcap_{i<n} b_i \leqq a + b_i \quad \text{for every} \quad i < n.$$

If

$$x \leqq a + b_i \quad \text{for} \quad i < n,$$

we conclude from 2.29 that for some b

$$x \leqq a + b, \quad \text{and} \quad b \leqq b_i \quad \text{for} \quad i < n,$$

and consequently

$$x \leqq a + \bigcap_{i<n} b_i .$$

Hence the conclusion.

THEOREM 3.26. *If* $0 < n \leqq \infty$ *and* $\bigcup_{i<n} b_i$ *exists, then*

$$a + \bigcup_{i<n} b_i = \bigcup_{i<n} (a + b_i).$$

PROOF: analogous to that of 3.25, with 2.29 replaced by 2.26.

THEOREM 3.27. *If* $m \leqq \infty$, $n \leqq \infty$, *but neither* $m = n = 0$ *nor* $m = n = \infty$, *and if* $\bigcap_{i<n} a_i$ *exists, then*

$$m \cdot \bigcap_{i<n} a_i = \bigcap_{i<n} (m \cdot a_i).$$

PROOF: analogous to that of 3.25, 2.29 being replaced by 2.38 (with $d = 0$).

Regarding the case of $m = n = \infty$, cf. the remark which follows 2.38.

THEOREM 3.28. *If* $n \leqq \infty$, $m \leqq \infty$, *and* $\bigcup_{i<n} a_i$ *exists, then*

$$m \cdot \bigcup_{i<n} a_i = \bigcup_{i<n} (m \cdot a_i).$$

PROOF: analogous to 3.25, with 2.29 replaced by 2.39 (the case $n = 0$ being obvious).

Each of the last two distributive laws, 3.30 and 3.32, is preceded by a lemma, 3.29 and 3.31.

LEMMA 3.29. *If $n \leqq \infty$, $c \leqq a \cup b_i$ for every $i < n$, $a \leqq d$, and*

$$\bigcap_{i<n} b_i \leqq d,$$

then $c \leqq d$.

PROOF: By hypothesis we can put

$$e + \bigcap_{i<n} b_i = d;$$

hence, by 3.25,

$$(1) \qquad \bigcap_{i<n} (e + b_i) = d.$$

Consequently, in view of the hypothesis,

$$a \leqq e + b_i$$

and

$$(2) \qquad c \leqq a \cup b_i \leqq (e + b_i) \cup b_i = e + b_i \quad \text{for every} \quad i < n.$$

The conclusion follows at once from (1) and (2).

THEOREM 3.30. *If $n \leqq \infty$, and if all the elements $a \cup b_i$ with $i < n$ exist, as well as $\bigcap_{i<n} b_i$, and either $a \cup \bigcap_{i<n} b_i$ or $\bigcap_{i<n} (a \cup b_i)$, then*

$$a \cup \bigcap_{i<n} b_i = \bigcap_{i<n} (a \cup b_i).$$

PROOF: Assume, e.g., that

$$a \cup \bigcap_{i<n} b_i$$

exists. We have by 3.1 and 3.2

$$a \cup \bigcap_{i<n} b_i \leqq a \cup b_i \quad \text{for every} \quad i < n;$$

and from 3.29 we conclude:

$$\text{if} \quad x \leqq a \cup b_i \quad \text{for every} \quad i < n, \quad \text{then} \quad x \leqq a \cup \bigcap_{i<n} b_i .$$

Hence, by 3.1, the conclusion. Analogously under the alternative assumption.

LEMMA 3.31. *If* $n \leqq \infty$, $a \cap b_i \leqq c$ *for every* $i < n$, $d \leqq a$, *and*

$$d \leqq \bigcup_{i<n} b_i,$$

then $d \leqq c$.

PROOF: There are elements $x_0, x_1, \cdots, x_i, \cdots, y_0, y_1, \cdots, y_i, \cdots$ such that, for every $i < n$,

$$x_i + (a \cap b_i) = a \tag{1}$$

and

$$y_i + (a \cap b_i) = b_i. \tag{2}$$

From (1), (2), and the hypothesis we obtain in an elementary way first

$$d \leqq x_i + c \quad \text{for} \quad i < n; \tag{3}$$

then

$$b_i \leqq \sum_{i<n} y_i + c \quad \text{for} \quad i < n;$$

and hence

$$d \leqq \sum_{i<n} y_i + c. \tag{4}$$

By applying 2.29 to (3) and (4), we get an element e such that

$$d \leqq e + c, \quad e \leqq x_i \quad \text{for} \quad i < n, \quad \text{and} \quad e \leqq \sum_{i<n} y_i. \tag{5}$$

An application of 2.2 to the last inequality gives

$$e = \sum_{i<n} e_i \quad \text{with} \quad e_i \leqq y_i \quad \text{for} \quad i < n. \tag{6}$$

(1), (2), (5), and (6) imply

$$e_i + (a \cap b_i) \leqq a, \quad e_i + (a \cap b_i) \leqq b_i;$$

consequently,

$$e_i + (a \cap b_i) = a \cap b_i,$$

and hence, by 1.28 and the hypothesis,

(7) $$e_i + c = c \quad \text{for} \quad i < n.$$

From (6) and (7) we obtain by 1.47

$$e + c = c;$$

and this formula together with (5) gives the conclusion.

With some later applications in view (see the proof of Theorem 11.28 below), we give here another, more elementary, proof of 3.31, which applies, however, only to the case when $n < \infty$. We consider, as before, the elements y_i which satisfy (2), and we obtain (4). Furthermore, we put, in view of the hypothesis,

(8) $$z_i + (a \cap b_i) = c \quad \text{for} \quad i < n.$$

(2) and (8) imply

(9) $$y_i + c = z_i + b_i \quad \text{for} \quad i < n.$$

Our theorem is obvious in case $n = 0$. If $n \neq 0$, we obtain from (4) and (9)

$$d \leqq \sum_{i<n-1} y_i + z_{n-1} + b_{n-1};$$

hence, by 3.3 and the hypothesis,

$$d \leqq \sum_{i<n-1} y_i + z_{n-1} + (a \cap b_{n-1});$$

and finally by (8)

(10) $$d \leqq \sum_{i<n-1} y_i + c.$$

Thus, we have derived (10) from (4). By continuing this procedure we obtain

$$d \leqq \sum_{i<m} y_i + c \quad \text{for every} \quad m < n;$$

and, by putting here $m = 0$, we arrive at the conclusion.

THEOREM 3.32. *If $n \leqq \infty$, and if all the elements $a \cap b_i$ with $i < n$ exist, as well as $\bigcup_{i<n} b_i$, and either $a \cap \bigcup_{i<n} b_i$ or $\bigcup_{i<n}(a \cap b_i)$, then*

$$a \cap \bigcup_{i<n} b_i = \bigcup_{i<n} (a \cap b_i).$$

PROOF: by 3.1, 3.2, and 3.31.

We omit here various elementary consequences and extensions of distributive laws, which are familiar from the discussion of other algebraic systems.

THEOREM 3.33. *If $n \leqq \infty$, $b_i \cap b_j = 0$ for all i and j with $i < j < n$, and either the element $a \cap \sum_{i<n} b_i$ or all the elements $a \cap b_i$ with $i < n$ exist, then all the elements involved exist, and*

$$a \cap \sum_{i<n} b_i = \sum_{i<n} (a \cap b_i).$$

PROOF: If $a \cap \sum_{i<n} b_i$ exists, then all the elements $a \cap b_i$ exist by 3.13. If all elements $a \cap b_i$ with $i < n$ exist, we apply 3.23 to the sequences $b_0, b_1, \cdots, b_i, \cdots$ and $a \cap b_0, a \cap b_1, \cdots, a \cap b_i, \cdots$; and by means of 3.32 we obtain the conclusion.

In 3.11, 3.14, and 3.33 we have come across various particular cases of the distributive law

$$a \cap \sum_{i<n} b_i = \sum_{i<n} (a \cap b_i).$$

In its general form, however—i.e., without restrictions regarding the elements involved—this law fails; as is easily seen, if it were true (even in case $n = 2$), it would imply that $a = 2 \cdot a$ for every a. A similar remark applies to the analogous law obtained by changing '$\cap$' to '$\cup$'.

With the help of 3.32 we can partially generalize 3.4 and 3.21:

THEOREM 3.34. *If $n \leqq \infty$, and if $\bigcap_{i<p} a_{k_i}$ exists for every p-termed sequence $k_0, k_1, \cdots, k_i, \cdots$ with $0 < p < \infty$ and $k_i < n$ for $i < p$, then $\bigcup_{i<n} a_i$ exists.*

PROOF: In case $n < \infty$ we proceed by induction. The theorem being obvious for $n = 0, 1$, we assume it to hold for a given $n = k$, $1 \leqq k < \infty$; and we assume its hypothesis to be satisfied by some elements $a_0, a_1, \cdots, a_i, \cdots$ with $i < k + 1$. We conclude that

$$\bigcup_{i<k} a_{i+1} \quad \text{and} \quad \bigcup_{i<k} (a_0 \cap a_{i+1})$$

exist. Hence, by 3.32,

$$a_0 \cap \bigcup_{i<k} a_{i+1}$$

exists; and, by applying 3.4, we obtain the conclusion for $n = k + 1$. By 3.22, the result can be extended at once to the case $n = \infty$.

We conclude this section with two rather special theorems regarding the existence of bounds. The notions of a WELL ORDERED set and of an AT MOST DENUMERABLE set, which occur in these two theorems, are assumed to be known.

THEOREM 3.35. *Let* $a_0, a_1, \cdots, a_i, \cdots$ *be any elements with* $i < n \leqq \infty$, *and let* B *be the set of all elements* x *such that* $x \leqq a_i$ *for every* $i < n$. *If every subset of* B *which is well ordered by the relation* $\leqq$ *is at most denumerable, then* $\bigcap_{i<n} a_i$ *exists.*

PROOF: By means of the well-ordering principle and in view of the hypothesis we can construct an infinite sequence of elements $b_0, b_1, \cdots, b_j, \cdots$ in B such that

$$b_j \leqq b_{j+1} \quad \text{for} \quad j < \infty,$$

and such that there is no x in B with

$$b_j \leqq x \quad \text{and} \quad b_j \neq x \quad \text{for} \quad j = 0, 1, 2, \cdots .$$

By 2.28 (or 3.21) we obtain an element b with

$$b_j \leqq b \leqq a_i \quad \text{for} \quad i < n \quad \text{and} \quad j < \infty;$$

and we easily see that b satisfies conditions (i) and (ii) of 3.16. Hence

$$\bigcap_{i<n} a_i = b.$$

THEOREM 3.36. *Let* $a_0, a_1, \cdots, a_i, \cdots$ *be any elements with* $i < n \leqq \infty$, *and let* B *be the set of all elements* x *such that* $a_i \leqq x$ *for every* $i < n$. *If every subset of* B *which is well ordered by the relation* $\geqq$ *is at most denumerable, then* $\bigcup_{i<n} a_i$ *exists.*

PROOF: analogous to that of 3.35, with 3.16 replaced by 3.17.

§ 4. SPECIAL KINDS OF ELEMENTS

We want to discuss now certain important kinds of elements in a C.A.

DEFINITION 4.1. *An element* $a \varepsilon A$ *is called* IDEM-MULTIPLE *if* $a + a = a$.

COROLLARY 4.2. 0 *is idem-multiple.*
PROOF: by 4.1.

THEOREM 4.3. *If* $1 < n \leqq \infty$ *and* $0 < p < q \leqq \infty$, *then the following conditions are equivalent:*

(i) $\qquad a$ *is idem-multiple*;

(ii) $$n \cdot a = a;$$

(iii) $$p \cdot a = q \cdot a.$$

PROOF: Conditions (i) and (ii) are equivalent by 1.46 and 4.1. Hence (i) implies (iii); and from (iii) we derive (ii) by means of 1.46 and 2.12.

THEOREM 4.4. *If* a *is idem-multiple, then, for every* b, *the formulas* $a \leqq b$ *and* $a + b = b$ *are equivalent, and so are the formulas* $b \leqq a$ *and* $b + a = a$.
PROOF: by 1.28, 1.30, and 4.1.

THEOREM 4.5. *For every element* a, $\infty \cdot a$ *is idem-multiple; if* b *is idem-multiple and* $a \leqq b$, *then* $\infty \cdot a \leqq b$.
PROOF: by 1.29, 1.43, 4.1, and 4.4.

THEOREM 4.6. *If at least one of the elements* a *and* b *is idem-multiple, then* $a + b = a \cup b$.
PROOF: by 3.6 (or 3.2) and 4.4.

THEOREM 4.7. *If* $n \leqq \infty$, *and all the elements* a_i *with* $i < n$ *are idem-multiple, then* $\sum_{i<n} a_i$ *and* $\bigcup_{i<n} a_i$ *are idem-multiple, and*

$$\sum_{i<n} a_i = \bigcup_{i<n} a_i.$$

PROOF: $\sum_{i<n} a_i$ is clearly idem-multiple by 4.1 (and 1.45). If

$$a_i \leqq x \text{ for every } i < n,$$

we obtain by 4.4 (and 1.47)

$$\sum_{i<n} a_i \leqq x.$$

Hence, and by 3.2, the formula of the conclusion follows at once.

THEOREM 4.8. *If $n \leqq \infty$, $a_i \cap a_j = 0$ for $i < j < n$, and $\sum_{i<n} a_i$ is idem-multiple, then all the elements a_i with $i < n$ are idem-multiple.*

PROOF: We apply 3.15 with

$$a = \sum_{i<n} a_i \quad \text{and} \quad b_i = a_i;$$

by 4.1 we obtain directly the conclusion.

THEOREM 4.9. *If $0 < n \leqq \infty$, $\bigcap_{i<n} a_i$ exists, and all the elements a_i with $i < n$ are idem-multiple, then $\bigcap_{i<n} a_i$ is idem-multiple.*

PROOF: by 4.1, and either 3.1 or 3.27 (with $m = 2$).

DEFINITION 4.10. *An element $a \varepsilon A$ is called* FINITE *if, for every $x \varepsilon A$, $x + a = a$ implies $x = 0$; otherwise, it is called* INFINITE.

COROLLARY 4.11. *Every idem-multiple element $a \neq 0$ is infinite.*

PROOF: by 4.1 and 4.10.

COROLLARY 4.12. *If $a \neq 0$, then $\infty \cdot a$ is infinite.*

PROOF: by 4.5 and 4.11.

COROLLARY 4.13. *0 is finite.*

PROOF: by 4.10.

COROLLARY 4.14. *If a is finite and $b \leqq a$, then b is finite.*

PROOF: by 1.28 and 4.10.

COROLLARY 4.15. *If $0 < n \leqq \infty$, $\bigcap_{i<n} a_i$ exists, and at least one of the elements a_i with $i < n$ is finite, then $\bigcap_{i<n} a_i$ is finite.*

PROOF: by 4.14.

THEOREM 4.16. *If $n < \infty$, and all the elements a_i with $i < n$ are finite, then $\sum_{i<n} a_i$ is finite.*

PROOF: If

$$x + \sum_{i<n} a_i = \sum_{i<n} a_i,$$

we apply 2.19 and, by means of 4.10, we obtain

$$x = 0.$$

(The same conclusion can be obtained by induction from Theorem 4.19 below, whose proof does not involve 4.16.) Hence, by 4.10, $\sum_{i<n} a_i$ is finite.

Corollary 4.17. *If $n < \infty$, $\bigcup_{i<n} a_i$ exists, and all the elements a_i with $i < n$ are finite, then $\bigcup_{i<n} a_i$ is finite.*

Proof: by 3.18, 4.14, and 4.16.

Theorem 4.18. *If $n \leqq \infty$, $a_i \cap a_j = 0$ for all i and j with $i < j < n$, and all the elements a_i with $i < n$ are finite, then $\sum_{i<n} a_i$ is finite.*

Proof: analogous to 4.16, with 2.19 replaced by 3.15.

Theorem 4.19.[13] *The following three conditions are equivalent:*
(i) *c is finite;*
(ii) *$a + c = b + c$ implies $a = b$ for any a and b:*
(iii) *$a + c \leqq b + c$ implies $a \leqq b$ for any a and b.*

Proof: From (i) we derive (iii) by means of 2.9 and 4.10; (iii) clearly implies (ii); and, by putting $b = 0$ in (ii) and applying 4.10, we obtain (i).

Thus, the cancellation laws for sums prove to hold for arbitrary finite elements of a C.A.

Corollary 4.20. *If $a + c = b + d$, $c \leqq d$, and c is finite, then $b \leqq a$.*

Proof: by 4.19.

Theorem 4.21. *Let $0 < n \leqq \infty$, and let c be finite. If $a_i + b_i = c$ for every $i < n$, and either $\bigcap_{i<n} a_i$ or $\bigcup_{i<n} b_i$ exists, then both these elements exist, and*

$$\bigcap_{i<n} a_i + \bigcup_{i<n} b_i = c.$$

Proof: Assume, e.g., that $\bigcup_{i<n} b_i$ exists. We then have, by 3.2 and the hypothesis,

$$\bigcup_{i<n} b_i \leqq c.$$

Let

$$d + \bigcup_{i<n} b_i = c.$$

[13] This theorem (originating with the author) is stated, in its application to cardinal numbers, in Lindenbaum-Tarski [1], p. 302.

We easily show, by means of 3.1, 4.14, and 4.20, that

$$d = \bigcap_{i<n} a_i.$$

Hence the conclusion. To obtain the same result under the assumption that $\bigcap_{i<n} a_i$ exists, we have to apply, in addition, 3.17.

COROLLARY 4.22. *If a and b are finite and $a \cup b$ exists, then $a \cap b$ also exists and $(a \cap b) + (a \cup b) = a + b$.*

PROOF: by 4.16 and 4.21 (with $n = 2$, $a_0 = b_1 = a$, and $b_0 = a_1 = b$).

COROLLARY 4.23. *If a and b are finite, and $a \cup b = a + b$, then $a \cap b = 0$.*

PROOF: by 4.10, 4.16, and 4.22.

THEOREM 4.24. *If $a_{n+1} \leqq a_n$ for every $n < \infty$, and at least one of the elements a_n with $n < \infty$ is finite, then $\bigcap_{i<n} a_i$ exists.*

PROOF: We can obviously assume without loss of generality that a_0 is finite. We have, by hypothesis, for some elements b_0, b_1, $\cdots$, b_n, $\cdots$,

$$a_n + b_n = a_{n+1} + b_{n+1} = a_0 \quad \text{for every} \quad n < \infty.$$

Hence, by 4.13 and 4.20

$$b_n \leqq b_{n+1} \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

Consequently, $\bigcup_{i<\infty} b_i$ exists by 3.21, and $\bigcap_{i<\infty} a_i$ exists by 4.21.

COROLLARY 4.25. *If $\bigcap_{i<n} a_i$ exists for every n with $0 < n < \infty$, and at least one of the elements a_i with $i < \infty$ is finite, then $\bigcap_{i<\infty} a_i$ exists and*

$$\bigcap_{i<\infty} a_i = \bigcap_{n<\infty} \bigcap_{i<n+1} a_i.$$

PROOF: by 4.15 and 4.24.

By means of 3.30 and 4.25, we could now generalize 4.22 in the same way in which 3.4 was generalized in 3.34.

DEFINITION 4.26. *An element $a \in A$ is called* MULTIPLE-FREE *if, for every $x \in A$, $2 \cdot x \leqq a$ implies $x = 0$.*

The meaning of this definition would perhaps be clearer if we used a multiplicative terminology; it would then be seen that mul-

tiple-free elements correspond to what are called in number theory 'square-free' integers, i.e., integers which are not divisible by any square different from 1.

COROLLARY 4.27. *No multiple-free element $a \neq 0$ is idem-multiple; every multiple-free element is finite.*

PROOF: by 4.1, 4.10, and 4.26.

COROLLARY 4.28. *0 is multiple-free.*

PROOF: by 4.26.

COROLLARY 4.29. *If a is multiple-free and $b \leqq a$, then b is multiple-free.*

PROOF: by 4.26.

COROLLARY 4.30. *If $0 < n \leqq \infty$, $\bigcap_{i<n} a_i$ exists, and at least one of the elements a_i with $i < n$ is multiple-free, then $\bigcap_{i<n} a_i$ is multiple-free.*

PROOF: by 4.29.

COROLLARY 4.31. *If $n \leqq \infty$, a is multiple-free, and $a = n \cdot b$, then $a = 0$ or else $n = 1$ and $a = b$.*

PROOF: by 4.26.

THEOREM 4.32. *Let n be $\leqq \infty$. In order that $\sum_{i<n} a_i$ be multiple-free, it is necessary and sufficient that*

(i) $\qquad a_i \cap a_j = 0 \quad$ *for all i and j with* $\quad i < j < n$,

and

(ii) $\qquad$ *all the elements a_i with $i < n$ be multiple free.*

PROOF: Assume $\sum_{i<n} a_i$ to be multiple-free. If

$$i < j < n, \quad x \leqq a_i, \quad \text{and} \quad x \leqq a_j,$$

we easily conclude from 4.26 that

$$x = 0.$$

Hence (i) holds; (ii) follows from 4.29. Assume now that (i) and (ii) hold, and consider an element x with

$$(1) \qquad 2 \cdot x \leqq \sum_{i<n} a_i.$$

By applying successively 2.2 and 2.1, we obtain:

(2) $$x = \sum_{i<n} y_i = \sum_{i<n} z_i \quad \text{where} \quad y_i + z_i \leqq a_i \quad \text{for} \quad i < n.$$

By 3.13, (i), and (1), the formulas just given imply

$$y_i = x \cap a_i = z_i \quad \text{for} \quad i < n;$$

hence, by 4.26 and (ii), and with the help of (2),

$$y_i = 0,$$

so that finally

(3) $$x = 0.$$

Thus, (1) implies (3). Hence, by 4.26, $\sum_{i<n} a_i$ is multiple-free, and the proof is complete.

THEOREM 4.33. *If $n \leqq \infty$, $a \leqq n \cdot b$, and a is multiple-free, then $a \leqq b$.*

PROOF: By 2.2 we obtain

$$a = \sum_{i<n} a_i, \quad \text{and} \quad a_i \leqq b \quad \text{for} \quad i < n.$$

Hence, by 3.23 and 4.32 we conclude that

$$a = \bigcup_{i<n} a_i \quad \text{where} \quad a_i \leqq b \quad \text{for} \quad i < n;$$

and this immediately implies the conclusion.

THEOREM 4.34. *If $n \leqq \infty$, $\bigcup_{i<n} a_i$ exists and is finite, and all elements a_i with $i < n$ are multiple-free, then $\bigcup_{i<n} a_i$ is multiple-free.*

PROOF: Consider an x for which

(1) $$2 \cdot x \leqq \bigcup_{i<n} a_i.$$

Hence, for some y,

(2) $$2 \cdot x + y = \bigcup_{i<n} a_i;$$

therefore

$$a_i \leqq 2 \cdot x + y \leqq 2 \cdot (x + y) \quad \text{for} \quad i < n.$$

and consequently, by 4.33,

$$a_i \leqq x + y \quad \text{for} \quad i < n.$$

This gives

(3) $$\bigcup_{i<n} a_i \leqq x + y.$$

From (2) and (3) we easily obtain

$$x + \bigcup_{i<n} a_i = \bigcup_{i<n} a_i\,;$$

and hence, by 4.10 and the hypothesis,

(4) $$x = 0.$$

Thus, (1) implies (4); the conclusion follows by 4.26.

COROLLARY 4.35. *If* $n < \infty$, $\bigcup_{i<n} a_i$ *exists, and all elements* a_i *with* $i < n$ *are multiple-free, then* $\bigcup_{i<n} a_i$ *is multiple-free.*

PROOF: by 4.17, 4.27, and 4.34. To justify some later remarks (see the proof of Theorem 11.28), we notice that an application of 4.17 can be avoided here since we can show directly that $\bigcup_{i<n} a_i$ is finite. In fact, in view of 3.18 and 4.14, it suffices to show that $\sum_{i<n} a_i$ is finite; and this in turn reduces to proving that, for any a, b, and c,

(1) $$a + c = b + c$$

implies

(2) $$a = b$$

under the assumption that c is multiple-free. (Cf. here a remark in the proof of 4.16.) Now, from (1) we obtain by 2.3

(3) $$a = d_1 + d_2\,, \quad b = d_1 + d_3\,, \quad \text{and} \quad c = d_3 + d_4 = d_2 + d_4$$

for some elements d_1, d_2, d_3, and d_4. The last of these formulas implies, by 4.32 and 3.5,

$$d_2 \cap d_4 = d_3 \cap d_4 = 0 \quad \text{and} \quad d_2 \cup d_4 = d_3 \cup d_4 = c;$$

hence, by 3.8,

(4) $$d_2 = d_3\,;$$

and from (3) and (4) we obtain (2). The argument may seem somewhat complicated; actually, however, it is quite elementary, whereas the proof of 4.17 involves some deeper results of §2.

The problem remains open whether the assumption of finiteness in 4.34 is essential; i.e., whether or not 4.35 can be extended to the case when $n = \infty$. By means of 4.32 we can show, however, that the least upper bound of an increasing sequence of multiple-free elements is always multiple-free.

THEOREM 4.36. *If $n \leqq \infty$, $a_i \leqq b$ for every $i < n$, and b is multiple-free, then $\bigcup_{i<n} a_i$ exists and is multiple-free.*

PROOF: If $n < \infty$, we prove the existence of $\bigcup_{i<n} a_i$ by induction. The case $n = 0$ is obvious. If now the conclusion holds for a given n, we put

$$a_n + c = b.$$

We have, by the inductive premise and 4.32,

$$a_n \cap c = 0 \quad \text{and} \quad \bigcup_{i<n} a_i \leqq a_n + c.$$

Hence, by 3.13,

$$(\bigcup_{i<n} a_i) \cap a_n$$

exists; and, by applying 3.4, we conclude that $\bigcup_{i<n+1} a_i$ also exists. By 3.22 the result can be extended by induction to the case $n = \infty$. The last part of the conclusion follows from 4.29.

THEOREM 4.37. *If $0 < n \leqq \infty$ and if, for every i and j with $i \leqq j < n$, there is a multiple-free element b with $a_i \leqq b$ and $a_j \leqq b$, then $\bigcap_{i<n} a_i$ exists and is multiple-free.*

PROOF: In case $n = 1$ the theorem obviously holds by 4.29. In case $n = 2$ the theorem follows from 3.13, 4.29, and 4.32 (or else from 4.22, 4.27, 4.29, and 4.36). The result can be extended by induction to every finite n, and by 4.25, 4.27, and 4.29 to the case of $n = \infty$.

DEFINITION 4.38. *An element $a \in A$ is called* INDECOMPOSABLE *if $a \neq 0$ and if, for any x and y in A, $x + y = a$ implies that $x = 0$ or $y = 0$.*

The importance of the notion of an indecomposable element (and also of the notions of a finite and of a multiple-free element) is restricted by the fact that in many C.A.'s no such elements occur at all. There are also C.A.'s which contain only indecomposable elements

in a weaker sense, i.e., elements $a \neq 0$ for which $x + y = a$ implies that $x = 0$ or $x = a$; or in a still weaker sense—elements $a \neq 0$ for which $x + y = a$ implies that $x = a$ or $y = a$. We are not going to discuss here these weaker notions of indecomposable elements.

COROLLARY 4.39. *No indecomposable element is idem-multiple; every indecomposable element is finite and multiple-free.*

PROOF: by 4.1, 4.10, 4.26, and 4.38.

COROLLARY 4.40. *If a is indecomposable and $b \leqq a$, then $b = 0$ or $b = a$.*

PROOF: by 4.38.

COROLLARY 4.41. *If a is indecomposable, then, for every b, $a \leqq b$ or $a \cap b = 0$.*

PROOF: by 4.40.

COROLLARY 4.42. *If $n \leqq \infty$, a is indecomposable, and*

$$a = \sum_{i<n} b_i,$$

then there is one and only one number $i < n$ such that $b_i = a$, while $b_j = 0$ for every j with $j < n$ and $j \neq i$.

PROOF: by 4.38.

THEOREM 4.43. *If $n \leqq \infty$, a is indecomposable, and*

$$a \leqq \sum_{i<n} b_i,$$

then $a \leqq b_i$ for some $i < n$.

PROOF: by 2.2 and 4.42.

THEOREM 4.44. *If $p \leqq \infty$, the elements $b_0, b_1, \cdots, b_i, \cdots$ are indecomposable, and*

$$a \leqq \sum_{i<p} b_i,$$

then there is a number $n \leqq p$ and a sequence of integers $m_0, m_1, \cdots, m_i, \cdots$ such that

$$a = \sum_{i<n} b_{m_i}, \quad m_i < p \quad \textit{for} \quad i < n, \quad \textit{and} \quad m_i < m_j \quad \textit{for} \quad i < j < p.$$

PROOF: By 2.2 we have

$$a = \sum_{i<p} a_i \quad \text{where} \quad a_i \leqq b_i \quad \text{for} \quad i < p.$$

Hence, with the help of 4.40, the conclusion.

THEOREM 4.45. *If* $n \leqq \infty$, $p \leqq \infty$, *the elements* $a_0, a_1, \cdots, a_i, \cdots$ *and* $b_0, b_1, \cdots, b_j, \cdots$ *are indecomposable, and*

$$\sum_{i<n} a_i = \sum_{j<p} b_j,$$

then $n = p$, *and the sequences* $a_0, a_1, \cdots, a_i, \cdots$ *and* $b_0, b_1, \cdots, b_j, \cdots$ *differ at most in their order.*

PROOF: By 2.1 there is a double sequence of elements $c_{i,j}$ with

$$a_i = \sum_{j<p} c_{i,j} \quad \text{and} \quad b_j = \sum_{i<n} c_{i,j} \quad \text{for} \quad i < n \quad \text{and} \quad j < p.$$

By 4.42 we infer that for every $i < n$, or $j < p$, there is one and only one number $j < p$, or $i < n$, with

$$a_i = c_{i,j} = b_j.$$

Hence easily the conclusion.

THEOREM 4.46. *If* $n \leqq \infty$, $p \leqq \infty$, *the elements* $a_0, a_1, \cdots, a_i, \cdots$ *are different from* 0, *the elements* $b_0, b_1, \cdots, b_j, \cdots$ *are indecomposable, and*

$$\sum_{i<n} a_i \leqq \sum_{j<p} b_j,$$

then $n \leqq p$.

PROOF: By 4.44, the element $\sum_{i<n} a_i$ can be represented as a sum of m indecomposable elements with $m \leqq p$. By applying 4.44 again, we obtain a decomposition of each of the elements a_i into m_i indecomposable elements; since $a_i \neq 0$, we have also $m_i \neq 0$. Thus, in view of 1.42, $\sum_{i<n} a_i$ has been decomposed into l indecomposable elements where

$$l = \sum_{i<n} m_i \quad (\text{and } l = \infty \text{ in case } n = \infty).$$

Hence, with the help of 4.45,

$$n \leqq l = m \leqq p,$$

and this gives the conclusion at once.

THEOREM 4.47. *If* $n, l_0, l_1, \cdots, l_i, \cdots$ *are integers* $\leqq \infty$, $b_0, b_1, \cdots, b_i, \cdots$ *are indecomposable elements,* $b_i \neq b_j$ *for* $i < j < n$, *and*

$$a \leqq \sum_{i<n} (l_i \cdot b_i),$$

then there is a uniquely determined sequence of integers $k_0, k_1, \cdots, k_i, \cdots$ *with* $i < n$ *such that*

$$a = \sum_{i<n} (k_i \cdot b_i);$$

and we have $k_i \leqq l_i$ *for every* $i < n$.

PROOF: The existence of such a sequence results from 2.2 and 4.44; its unicity is a consequence of 1.42 and 4.45.

THEOREM 4.48. *If* $n \leqq \infty$, *and each of the elements* a_i *with* $i < n$ *can be represented in the form*

$$a_i = \sum_{j<p} b_j$$

where $p \leqq \infty$ *and the elements* $b_0, b_1, \cdots, b_j, \cdots$ *are indecomposable, then the element* $\sum_{i<n} a_i$ *can also be represented in this form. Moreover, the elements* $\bigcap_{i<n} a_i$ *(in case* $n \neq 0$*) and* $\bigcup_{i<n} a_i$ *exist and have also a similar representation.*

PROOF: The first part of the theorem is obvious (in view of 1.42). To derive the second part, we recall that the representations of elements a_i are unique by 4.45; and we arrange all elements occurring in these representations in a single sequence $c_0, c_1, \cdots, c_j, \cdots$ with $j < q \leqq \infty$, and without repeating terms. As is easily seen (from 1.42) we have then, for every $i < n$,

$$a_i = \sum_{j<q} (m_{i,j} \cdot c_j)$$

where $m_{i,j}$ are numbers $\leqq \infty$ (some of them may equal 0). Given a number $j < q$, let k_j be the smallest among the numbers $m_{i,j}$ with $i < n$; and let l_j be the least upper bound of the numbers $m_{i,j}$, i.e., the largest such number, or ∞ in case no largest number $m_{i,j}$ exists. We now show without difficulty that

$$\sum_{j<q} (k_j \cdot c_j) = \bigcap_{i<n} a_i \quad \text{and} \quad \sum_{j<q} (l_j \cdot c_j) = \bigcup_{i<n} a_i.$$

In fact, the first formula follows from 3.1 and 4.47, and the second from 3.17 and 4.47 (or else from 2.22, 3.2, 3.12, 3.23, 4.40, and 4.41). Hence the conclusion.

THEOREM 4.49. *Every element* a *satisfies one and only one of the following two conditions:*

(i) *there is an infinite sequence of elements* $b_0, b_1, \cdots, b_i, \cdots$ *different from* 0 *such that*

$$a = \sum_{i<\infty} b_i;$$

(ii) *there is a finite n-termed sequence of indecomposable elements* $c_0, c_1, \cdots, c_i, \cdots$ *such that*

$$a = \sum_{i<n} c_i.$$

PROOF: Assume a not to satisfy (ii). Then a must be $\neq 0$, and cannot be indecomposable; and hence we have by 4.38

$$a = x + y, \quad x \neq 0, \quad y \neq 0.$$

It is easily seen that at least one of the elements x and y, say y, does not satisfy (i); we put

$$a_0 = a, \quad b_0 = x, \quad a_1 = y.$$

By repeating this procedure infinitely many times, we arrive at two infinite sequences a_ι and b_ι with

$$a_n = b_n + a_{n+1}, \quad a_n \neq 0, \quad \text{and} \quad b_n \neq 0 \quad \text{for} \quad n = 0, 1, 2, \cdots;$$

and, by applying 1.1.VII, we obtain

$$a_0 = c + \sum_{i<\infty} b_i.$$

Hence we see that (i) holds. Thus, every element a satisfies one of the conditions (i) and (ii); and 4.46 implies that it cannot satisfy both these conditions.

With this we conclude the discussion of arithmetical properties of cardinal algebras.

PART II

METHODS OF CONSTRUCTION

§ 5. ELEMENTARY PROPERTIES OF GENERALIZED CARDINAL ALGEBRAS

The results established in Part I will now be extended to a wider class of algebraic systems which will be referred to as GENERALIZED CARDINAL ALGEBRAS. This will widen considerably the range of applications of the results obtained. Moreover, the new algebras have also a significance for the study of C.A.'s in themselves; for certain interesting methods of construction are known which lead from C.A.'s, not to new C.A.'s in the strict sense, but to generalized C.A.'s, while other methods enable us to obtain important examples of C.A.'s in the strict sense from algebras of the wider class.

A generalized cardinal algebra is again a system constituted by a set A of arbitrary elements and by two operations of addition, $+$ and $\sum$. However, the defining postulates differ essentially from those given in 1.1 by the lack of the general closure postulates, 1.1.I and 1.1.II. Thus, a sum of two elements of one of the new algebras need not belong to this algebra, and may not even exist at all; which of the two cases actually occurs is quite irrelevant for our discussion. At any rate, generalized cardinal algebras may be very 'imperfect' systems from an orthodox algebraic point of view.

DEFINITION 5.1. *An algebraic system $\mathfrak{A} = \langle A, +, \sum \rangle$ which satisfies the following postulates* I–V *is called a* GENERALIZED CARDINAL ALGEBRA, *for abbreviation, a G.C.A.* (*and under the same conditions the set A is said to be a G.C.A.* UNDER $+$ AND $\sum$):

I. *If $a_i \,\varepsilon\, A$ for every $i < \infty$ and $\sum_{i<\infty} a_i \,\varepsilon\, A$, then*

$$\sum_{i<\infty} a_{i+1} \,\varepsilon\, A \quad \textit{and} \quad \sum_{i<\infty} a_i = a_0 + \sum_{i<\infty} a_{i+1}.$$

II. *If a_i, b_i, $a_i + b_i \,\varepsilon\, A$ for every $i < \infty$ and $\sum_{i<\infty} (a_i + b_i) \,\varepsilon\, A$, then*

$$\sum_{i<\infty} a_i, \sum_{i<\infty} b_i \,\varepsilon\, A \quad \textit{and} \quad \sum_{i<\infty} (a_i + b_i) = \sum_{i<\infty} a_i + \sum_{i<\infty} b_i.$$

III. *Postulate* 1.1.V.

IV. *Postulate* 1.1.VI, *with the hypothesis supplemented by the condition that $a + b$ is in A.*

V. *Postulate* 1.1.VII, *with the conclusion supplemented by the condition that all sums* $\sum_{i<\infty} b_{n+i}$ *for* $n < \infty$ *are in* A.

We can extend to G.C.A.'s all the notions which have been defined in the arithmetic of C.A.'s:

DEFINITION 5.2. *Definitions* 1.2–1.5, 3.1, 3.2, 4.1, 4.10, 4.26, *and* 4.38 *apply without change to an arbitrary G.C.A. and, more generally, to an arbitrary algebra* $\mathfrak{A} = \langle A, +, \sum \rangle$.

Of course, the sum $\sum_{i<n} a_i$ of a finite sequence and the multiple $n \cdot a$, as defined in 1.3 and 1.4, are understood to exist only if the defining infinite sums exist. Definitions 3.1 and 3.2 will be applied in practice only to those algebras which are partially ordered by the relation $\leqq$.

Trivial examples of G.C.A.'s can be obtained in the following way. We consider an arbitrary non-empty set A and single out an element 0 in it. Furthermore, we define the operations $+$ and $\sum$ by putting:

$a + 0 = 0 + a = a$ for every $a \in A$, $\sum_{i<\infty} a_i = a_n$ in case $n < \infty$, $a_n \in A$, and $a_i = 0$ for every i with $i < \infty$ and $i \neq n$;

and by assuming that in no other case does the sum $a + b$ or $\sum_{i<\infty} a_i$ exist. The algebra $\mathfrak{A} = \langle A, +, \sum \rangle$ is clearly a G.C.A. in which every element different from 0 is indecomposable. We shall later come across much more interesting examples of G.C.A.'s.

We can now extend to G.C.A.'s, step by step, all the theorems established in Part I, by providing these theorems with additional assumptions which secure the existence of all or some of the sums involved. However, in view of the lack of the general closure postulates, the proofs require as a rule somewhat more care; and in more difficult cases (as, for instance, in the proof of 2.31) this defect may cause considerable complications. It is therefore fortunate that a result stated below in 7.8 will provide us with an automatic method of extending to G.C.A.'s arithmetical theorems which apply to arbitrary C.A.'s. This result, however, cannot be obtained without several preliminary steps. In the first place, a number of elementary theorems of Part I must be extended to G.C.A.'s directly. This is what we are going to do now.

In formulating the theorems just mentioned we shall, as before,

omit the assumptions by which the elements involved belong to the set A in a given G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$; however, the necessary assumptions regarding the existence of sums will always be stated explicitly.

THEOREM 5.3. *If* $\sum_{i<\infty} a_{i+1} \varepsilon A$ *and* $a_0 + \sum_{i<\infty} a_{i+1} \varepsilon A$, *then*

$$\sum_{i<\infty} a_i = a_0 + \sum_{i<\infty} a_{i+1}.$$

PROOF: By an easy induction we conclude from 5.1.I that

$$\sum_{i<\infty} a_{n+i+1} \varepsilon A \quad \text{and} \quad \sum_{i<\infty} a_{n+i+1} = a_{n+1} + \sum_{i<\infty} a_{n+i+2}$$

for every $n < \infty$. Hence, by putting

$$b_0 = a_0 + \sum_{i<\infty} a_{i+1} \quad \text{and} \quad b_{n+1} = \sum_{i<\infty} a_{n+i+1} \quad \text{for} \quad n = 0, 1, 2, \cdots,$$

we see that the hypothesis of 5.1.V, with a_i and b_i interchanged, is satisfied. Consequently,

$$\sum_{i<\infty} a_i \varepsilon A;$$

and, by applying 5.1.I, we at once obtain the conclusion.

THEOREM 5.4. $0 \varepsilon A$; *and we have* $a + 0 = 0 + a = a$ *for every* $a \varepsilon A$, *and* $n \cdot 0 = 0$ *for every* $n \leqq \infty$.

PROOF: practically the same as that of 1.6 and 1.7; 5.3 and 5.1.II, III, V are used instead of 1.1.III, IV, V, and VII.

COROLLARY 5.5. $a \leqq a$ *and* $0 \leqq a$ *for every* $a \varepsilon A$.

PROOF: by 5.2 (1.5) and 5.4.

THEOREM 5.6. $\sum_{i<0} a_i = 0$ *and* $\sum_{i<1} a_i = a_0$; *if* $\sum_{i<2} a_i \varepsilon A$ *or* $a_0 + a_1 \varepsilon A$, *then* $\sum_{i<2} a_i = a_0 + a_1$.

PROOF: The first two formulas follow directly from 5.2 (1.3, 1.4), 5.3, and 5.4. To derive the last formula, we put

$$a_i = 0 \quad \text{for} \quad i = 2, 3, \cdots.$$

Then, by the second formula and 5.2 (1.4), we have

$$\sum_{i<\infty} a_{i+1} = a_1.$$

Now, by using either 5.1.I or 5.3, we arrive at the conclusion.

THEOREM 5.7. *Theorem* 1.10 *holds in G.C.A.'s.*
PROOF: unchanged.

THEOREM 5.8. *If* $a + b \varepsilon A$, *then* $a + b = b + a$.
PROOF: by 5.1.II (with $a_1 = b$, $b_0 = a$, and $a_0 = a_{i+2} = b_{i+1} = 0$ for $i = 0, 1, 2, \cdots$), 5.2 (1.3), 5.4, and 5.6.

THEOREM 5.9. *If either* $a + b$ *and* $(a + b) + c$ *are in* A *or else* $b + c$ *and* $a + (b + c)$ *are in* A, *then all these sums are in* A, *and we have*

$$(a + b) + c = a + (b + c).$$

PROOF: analogous to that of 5.8 (or 1.14). For instance, under the first alternative we put in 5.1.II: $a_0 = a$, $b_0 = b$, $b_1 = c$, and $a_{i+1} = b_{i+2} = 0$ for $i = 0, 1, 2, \cdots$; and we apply 5.2, 5.4, and 5.6. The second alternative can be reduced to the first by means of 5.8.

COROLLARY 5.10. *If* $a \leqq b$ *and* $b \leqq c$, *then* $a \leqq c$.
PROOF: by 5.2 (1.5) and 5.9.

THEOREM 5.11. *If* $n < \infty$ *and* $\sum_{i<\infty} a_i \varepsilon A$, *then* $a_n + a_{n+1} \varepsilon A$.
PROOF: By a repeated application of 5.1.I we easily conclude that

$$\sum_{i<\infty} a_{n+i} = a_n + (a_{n+1} + \sum_{i<\infty} a_{n+i+2}) \quad \text{for every} \quad n < \infty,$$

all the sums involved being in A. The conclusion follows from the second alternative of 5.9.

THEOREM 5.12. *If* $n \leqq \infty$, *and either* $\sum_{i<n+1} a_i$ *is in* A, *or* $\sum_{i<n} a_{i+1}$ *and* $a_0 + \sum_{i<n} a_{i+1}$ *are in* A, *then all three sums are in* A, *and*

$$\sum_{i<n+1} a_i = a_0 + \sum_{i<n} a_{i+1}.$$

PROOF: by 5.2 (1.3), and 5.1.I or 5.3.

COROLLARY 5.13. *If* $\infty \cdot a \varepsilon A$, *then* $\infty \cdot a = a + \infty \cdot a = \infty \cdot a + a$.
PROOF: by 5.2 (1.4), 5.8, and 5.12.

THEOREM 5.14. *If* $n < \infty$, *and either* $\sum_{i<n+1} a_i$ *is in* A, *or* $\sum_{i<n} a_i$ *and* $\sum_{i<n} a_i + a_n$ *are in* A, *then all three sums are in* A, *and*

$$\sum_{i<n+1} a_i = \sum_{i<n} a_i + a_n.$$

PROOF: by induction based upon 5.4 (1.6), 5.6, 5.9, and 5.12.

Theorem 5.15. *If $n < \infty$, and either $\sum_{i<\infty} a_i$ is in A, or $\sum_{i<n} a_i$, $\sum_{i<\infty} a_{n+i}$, and $\sum_{i<n} a_i + \sum_{i<\infty} a_{n+i}$ are in A, then all these sums are in A, and we have*

$$\sum_{i<\infty} a_i = \sum_{i<n} a_i + \sum_{i<\infty} a_{n+i}.$$

Proof: by induction with respect to n (5.4, 5.6, 5.9, 5.12, and 5.14).

Theorem 5.16. *If $n < \infty$, and $a_i = 0$ for every i with $i < \infty$ and $i \neq n$, then*

$$\sum_{i<\infty} a_i = a_n .$$

Proof: By 5.2 (1.3, 1.4), 5.4, and 5.6 we have

$$\sum_{i<n} a_i = 0 \quad \text{and} \quad \sum_{i<\infty} a_{n+i} = a_n .$$

Hence, by 5.4 and 5.15, the conclusion.

Theorem 5.17. *If $a = c + b$ and $b = d + a$, then $a = b$.*
Proof: By 5.8 and 5.9 we obtain

$$a = (d + c) + a.$$

Hence, by 5.1.V and 5.2 (1.4), there is an element e with

$$a = e + \infty \cdot (d + c).$$

By applying 5.1.II (with $a_i = d$ and $b_i = c$), 5.8, and 5.9, we conclude that

(1) $$a = \infty \cdot d + (\infty \cdot c + e).$$

Hence, by 5.13 and 5.9,

(2) $$a = (d + \infty \cdot d) + (\infty \cdot c + e) = d + [\infty \cdot d + (\infty \cdot c + e)].$$

(1) and (2) give

$$a = d + a;$$

and this, together with the hypothesis of the theorem, at once implies the conclusion.

Corollary 5.18. *If $a \leqq b$ and $b \leqq a$, then $a = b$.*
Proof: by 5.2 (1.5), 5.8, and 5.17.

Thus, by 5.5, 5.10, and 5.18, the relation $\leqq$ establishes a partial order in every G.C.A.

COROLLARY 5.19. *If* $a + b = 0$, *then* $a = b = 0$.
PROOF: by 5.4 and 5.17.

COROLLARY 5.20. *If* $n < \infty$ *and* $\sum_{i<\infty} a_i = 0$, *then* $a_n = 0$.
PROOF: by 5.14 and 5.15 we obtain:

$$(\sum_{i<n} a_i + a_n) + \sum_{i<\infty} a_{n+i+1} = 0.$$

Hence, by 5.19, the conclusion.

THEOREM 5.21. *Theorem* 1.36 *holds in G.C.A.'s.*
PROOF: by 5.1.I, 5.1.V, 5.2 (1.5), 5.8, and 5.17.

THEOREM 5.22. *If* $\sum_{j<\infty} a_{i,j}$ *is in* A *for every* $i < \infty$ *and* $\sum_{i<\infty} \sum_{j<\infty} a_{i,j}$ *is in* A, *then also* $\sum_{i<\infty} a_{i,j}$ *is in* A *for every* $j < \infty$, *and we have*

$$\sum_{i<\infty} \sum_{j<\infty} a_{i,j} = \sum_{j<\infty} \sum_{i<\infty} a_{i,j}.$$

PROOF: By 5.15 we have for any given $n < \infty$

(1) $$\sum_{j<\infty} a_{i,j} = \sum_{j<n} a_{i,j} + \sum_{j<\infty} a_{i,n+j},$$

and by 5.1.I we obtain

(2) $$\sum_{j<\infty} a_{i,n+j} = a_{i,n} + \sum_{j<\infty} a_{i,n+j+1}.$$

Since $\sum_{i<\infty} \sum_{j<\infty} a_{i,j}$ is in A, we infer from (1) and (2) by applying 5.1.II twice, that

(3) $$\sum_{i<\infty} \sum_{j<\infty} a_{i,n+j} = \sum_{i<\infty} a_{i,n} + \sum_{i<\infty} \sum_{j<\infty} a_{i,n+j+1},$$

all the sums involved being in A. By putting

$$a_n = \sum_{i<\infty} \sum_{j<\infty} a_{i,n+j} \quad \text{and} \quad b_n = \sum_{i<\infty} a_{i,n} \quad \text{for} \quad n = 0, 1, 2, \cdots,$$

we conclude from (3) by 5.1.V that

$$\sum_{j<\infty} \sum_{i<\infty} a_{i,j} \,\varepsilon\, A$$

and that, for some c,

(4) $$\sum_{i<\infty} \sum_{j<\infty} a_{i,j} = c + \sum_{j<\infty} \sum_{i<\infty} a_{i,j}.$$

The existence of all sums involved being now secured, we obtain in an entirely analogous way

$$(5) \qquad \sum_{j<\infty} \sum_{i<\infty} a_{i,j} = d + \sum_{i<\infty} \sum_{j<\infty} a_{i,j}.$$

The conclusion follows from (4) and (5) by 5.17.

THEOREM 5.23. *Theorem* 2.1 *holds in G.C.A.'s (under the assumption that* $\sum_{i<n} a_i$ *is in* A*).*

PROOF: virtually unchanged.

These are all the arithmetical theorems which we need for the time being.

A G.C.A. which satisfies both closure postulates 1.1.I, II is obviously a C.A. Notice in this connection the following:

THEOREM 5.24. *For an algebra* $\mathfrak{A} = \langle A, +, \sum \rangle$ *to be a C.A. it is necessary and sufficient that* $\mathfrak{A}$ *be a G.C.A. and that it satisfy Postulate* 1.1.II.

PROOF: If $\mathfrak{A}$ is a G.C.A. which satisfies 1.1.II, it follows from 5.2 (1.3), 5.4, and 5.6 (last part), that it also satisfies 1.1.I, and hence is a C.A. The converse is obvious.

THEOREM 5.25. *For an algebra* $\mathfrak{A} = \langle A, +, \sum \rangle$ *to be a C.A. it is necessary and sufficient that* $\mathfrak{A}$ *be a G.C.A. and that, for any elements* $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$ *with* $i < \infty$, *there exists an idem-multiple element* $b \varepsilon A$ *such that* $a_i \leqq b$ *for* $i = 0, 1, 2, \cdots$.

PROOF: The necessity of the conditions is almost obvious; in view of 4.5, we can put

$$b = \infty \cdot \sum_{i<\infty} a_i.$$

Now assume these conditions to be satisfied. Let $a_0, a_1, \cdots, a_i, \cdots$ be any elements in A. Then there is an idem-multiple element $b \varepsilon A$ such that

$$a_i \leqq b,$$

and hence, for some elements $c_0, c_1, \cdots, c_i, \cdots \varepsilon A$, we have

$$a_i + c_i = b \quad \text{for} \quad i = 0, 1, 2, \cdots.$$

Since, by 5.2 (4.1),

$$b = b + b,$$

we conclude from 5.1.V (with $a_i = b_i = b$) that

$$\sum_{i<\infty} b = \infty \cdot b \, \varepsilon \, A.$$

Hence

$$\sum_{i<\infty} (a_i + c_i) = \infty \cdot b \, \varepsilon \, A,$$

and therefore, by 5.1.II, $\sum_{i<\infty} a_i$ is in A. Thus, the G.C.A. $\mathfrak{A}$ satisfies 1.1.I; and consequently, by 5.24, it is a C.A. This completes the proof.

We shall sometimes be concerned with G.C.A.'s which satisfy 1.1.I. without necessarily satisfying 1.1.II.

DEFINITION 5.26. *An algebra* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is said to be* FINITELY CLOSED *if it satisfies Postulate* 1.1.I.

COROLLARY 5.27. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a finitely closed G.C.A.,* $n < \infty$, *and* $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$, *then*

$$\sum_{i<n} a_i \, \varepsilon \, A.$$

PROOF: by 5.4, 5.6, 5.14, and 5.26.

§6. GENERAL METHODS OF CONSTRUCTION

In our further discussion we shall frequently be concerned in the same definition or theorem with two or more different algebras. Sometimes we shall use the same symbols, '$+$' and '$\sum$', when referring to fundamental operations of different algebras. This will be avoided, however, in cases in which it might easily result in confusion; we shall then employ different symbols for different algebras. For instance, we shall use the fundamental operation symbols provided with an apostrophe, a subscript, etc.; and in such a case other symbols which have been introduced in Part I, like '0', '$\leq$', '$\mathbf{U}$', etc., will also be provided with the same apostrophe or subscript.

We can apply to C.A.'s and G.C.A.'s various notions which are familiar from general algebra, like those of ISOMORPHISM, HOMOMORPHISM, and CARDINAL (or DIRECT) PRODUCT. As is well known, certain general methods of constructing new algebras from given ones are correlated with the notions just mentioned. By means of these methods we shall be able to establish in the next section a fundamental theorem by which every G.C.A. can be imbedded in a C.A. Since the methods in question are to be applied to rather 'unorthodox' algebraic systems, the definitions of all relevant notions will be formulated here explicitly. We are not going, however, to state those consequences of these definitions in which no specific properties of C.A.'s and G.C.A.'s are involved.

We begin with the notion of isomorphism. We shall speak of functions f which map an algebra $\mathfrak{A}$ isomorphically onto another algebra $\mathfrak{A}'$ (or transform $\mathfrak{A}$ isomorphically into $\mathfrak{A}'$). A function f is called BIUNIQUE if $f(x) = f(y)$ always implies $x = y$. The DOMAIN of f, i.e., the set of argument values, will be denoted by '$D(f)$', and the COUNTER-DOMAIN of f, i.e., the set of function values, by '$C(f)$'.

DEFINITION 6.1. *A function f is said to* MAP AN ALGEBRA $\mathfrak{A} = \langle A, +, \sum\rangle$ ISOMORPHICALLY ONTO AN ALGEBRA $\mathfrak{A}' = \langle A', +', \sum'\rangle$ *if it satisfies the following conditions:*

(i) *f is biunique;*

(ii) *$D(f) = A$ and $C(f) = A'$;*

(iii) *for any a, b, $c \in A$ we have*

$$a = b + c \text{ if, and only if, } f(a) = f(b) +' f(c);$$

(iv) *for any* $a, a_0, a_1, \cdots, a_i, \cdots \varepsilon A$ *we have*

$$a = \sum_{i<\infty} a_i \text{ if, and only if, } f(a) = \sum_{i<\infty}' f(a_i).$$

If such a function f exists, the algebras $\mathfrak{A}$ *and* $\mathfrak{A}'$ *are called* ISOMORPHIC, *in symbols,*

$$\mathfrak{A} \cong \mathfrak{A}'.$$

It goes without saying that an algebra $\mathfrak{A}'$ which is isomorphic with a C.A., or a G.C.A., $\mathfrak{A}$ is itself a C.A., or a G.C.A.; and that an isomorphic mapping preserves all 'intrinsic' properties of an algebra and of elements of this algebra.

THEOREM 6.2. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *and* $\mathfrak{A}' = \langle A', +', \sum' \rangle$ *are G.C.A.'s (or, in particular, C.A.'s), and if f is a function which satisfies conditions* (ii) *and* (iii), *or* (ii) *and* (iv), *of* 6.1, *then f maps* $\mathfrak{A}$ *isomorphically onto* $\mathfrak{A}'$, *and hence*

$$\mathfrak{A} \cong \mathfrak{A}'.$$

PROOF: Assume f to satisfy (ii) and (iii) of 6.1. By (ii) and 5.4 there is an element z in A with $f(z) = 0'$. Hence, if $f(x) = f(y)$, we have

$$f(x) = f(z) +' f(y) \quad \text{and} \quad f(y) = f(z) +' f(x);$$

and therefore, by (iii) and 5.17,

$$x = z + y, \quad y = z + x, \quad \text{and} \quad x = y.$$

Thus, f satisfies 6.1(i). Finally, in view of 5.21, the operation $\sum$ in G.C.A.'s can be defined in terms of $+$; hence, conditions (i)–(iii) of 6.1 imply condition (iv), and the function f maps $\mathfrak{A}$ isomorphically onto $\mathfrak{A}'$. The proof under the assumptions that f satisfies 6.1(ii), (iv) is essentially the same; in addition to 5.4 and 5.17, we apply 5.1.I, 5.6, and 5.8, while 5.21 is replaced by 5.7.

The definition of HOMOMORPHIC MAPPING can be obtained from 6.1 by omitting condition (i), and by slightly modifying (iii) and (iv). We are not going to formulate this definition, however, since in this work we shall make use of the well-known fact that the discussion of homomorphic images of an algebra $\mathfrak{A}$ can be replaced by that of the so-called COSET (or RESIDUE CLASS) ALGEBRAS of $\mathfrak{A}$ generated by a binary relation R.

DEFINITION 6.3. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$, *and let* R *be a binary relation. The* COSET a/R OF AN ELEMENT $a \varepsilon A$ *is the set of all elements* $x \varepsilon A$ *with* $a \; R \; x$. *By the* SUM $a/R + b/R$ OF TWO COSETS *we understand the coset* c/R *which contains all elements* $c' \varepsilon A$ *such that* $a' + b' = c'$, $a \; R \; a'$, *and* $b \; R \; b'$ *for some* a', $b' \varepsilon A$ *(provided such a coset* c/R *exists and is unique); similarly we define the* SUM $\sum_{i<\infty} (a_i/R)$ OF AN INFINITE SEQUENCE OF COSETS. *The algebra constituted by the family* C *of all cosets and by the operations* $+$ *and* $\sum$ *just defined is called the* COSET (*or* RESIDUE CLASS) ALGEBRA OF $\mathfrak{A}$ UNDER R, *in symbols,*

$$\mathfrak{A}/R = \langle \mathsf{C}, +, \textstyle\sum \rangle.$$

This construction will be applied exclusively in those cases when R is an EQUIVALENCE RELATION in $\mathfrak{A}$—i.e., a relation which is REFLEXIVE in A ($x \; R \; x$ for every $x \varepsilon A$), SYMMETRIC, and TRANSITIVE—and has certain conservative properties with respect to the fundamental operations of $\mathfrak{A}$. We shall refer to these properties as FINITE and INFINITE ADDITIVITY.

DEFINITION 6.4. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$. *A binary relation* R *is said to be*

(i) *FINITIVELY ADDITIVE IN* $\mathfrak{A}$,

or

(ii) *INFINITELY ADDITIVE IN* $\mathfrak{A}$,

if, for any $a, a_0, a_1, a_2, \cdots, b, b_0, b_1, b_2, \cdots \varepsilon A$, *the formulas*

(i) $\qquad a = a_1 + a_2, \quad b = b_1 + b_2, \quad a_1 R \; b_1, \quad$ *and* $\quad a_2 R \; b_2,$

or

(ii) $a = \sum_{i<\infty} a_i, \quad b = \sum_{i<\infty} b_i, \quad$ *and* $\quad a_i \; R \; b_i \quad$ *for* $\quad i = 0, 1, 2, \cdots,$

imply

$$a \; R \; b.$$

THEOREM 6.5. *Every relation* R *which is reflexive and infinitely additive in a G.C.A.* $\mathfrak{A}$ *is also finitely additive in* $\mathfrak{A}$.

PROOF: by extending couples to infinite sequences by means of zeros (cf. 5.6, last part).

As is well known, the coset algebra $\mathfrak{A}/R$ generated by an equivalence relation R which is finitely and infinitely additive in $\mathfrak{A} =$

$\langle A, +, \sum \rangle$ is always a homomorphic image of $\mathfrak{A}$; and, conversely, every homomorphic image of $\mathfrak{A}$ is isomorphic with a coset algebra $\mathfrak{A}/R$ of this kind. It is easily seen, however, that a coset algebra $\mathfrak{A}/R$ of a C.A., or a G.C.A., $\mathfrak{A}$ is not necessarily a C.A., or a G.C.A., itself. We have only:

THEOREM 6.6. *Let R be an infinitely additive equivalence relation in a C.A., or G.C.A., $\mathfrak{A}$. For $\mathfrak{A}/R$ to be a C.A., or G.C.A., it is sufficient (and necessary) that $\mathfrak{A}/R$ satisfy Postulates* 1.1.VI, VII, *or* 5.1.IV, V. *The zero element of $\mathfrak{A}/R$ is $0/R$.*

PROOF: With the help of 6.3–6.5 we easily check that $\mathfrak{A}/R$ satisfies the remaining postulates, i.e., 1.1.I–V, or 5.1.I–III; Postulate 1.1.V, or 5.1.III, is clearly satisfied by the coset $0/R$ (cf. 5.4).

From 6.6 we could derive a necessary and sufficient condition for $\mathfrak{A}/R$ to be a C.A., or a G.C.A.—a condition which would be expressed exclusively in terms of elements of A, the operations $+$ and $\sum$ on these elements, and the relation R. This criterion, however, would be rather involved (especially as applied to G.C.A.'s) and of little practical value. We shall succeed below (in 6.10) in establishing a simple, useful, and general criterion which provides a sufficient, though not necessary, condition for $\mathfrak{A}/R$ to be a C.A., or a G.C.A.

DEFINITION 6.7. *Let $\mathfrak{A} = \langle A, +, \sum \rangle$.*

(i) *A binary relation R is said to be* FINITELY REFINING IN $\mathfrak{A}$ *if, for all $a, a_1, a_2, b \in A$, the formulas*

$$a = a_1 + a_2 \quad \textit{and} \quad a\,R\,b, \quad \textit{or} \quad a = a_1 + a_2 \quad \textit{and} \quad b\,R\,a,$$

imply the existence of elements b_1 and b_2 such that

$$b = b_1 + b_2 \quad \textit{and} \quad a_i\,R\,b_i\,, \quad \textit{or}$$

$$b = b_1 + b_2 \quad \textit{and} \quad b_i\,R\,a_i\,, \qquad \textit{for } i = 1, 2.$$

(ii) *By changing in* (i) *sums of two elements to sums of infinite sequences, we arrive at the notion of an* INFINITELY REFINING *relation.*

COROLLARY 6.8. *Let R be a finitely, or infinitely, refining relation in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$. If $a, a', b \in A$, $a' \leqq a$, and $a\,R\,b$ (or $b\,R\,a$), then there is an element $b' \in A$ such that $b' \leqq b$ and $a'\,R\,b'$ (or $b'\,R\,a'$).*

PROOF: by 5.2 (1.3, 1.5), 5.4, 5.6, and 6.7.

THEOREM 6.9. *Every relation R which is infinitely additive and infinitely refining in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$ is also finitely refining in $\mathfrak{A}$.*

PROOF: If

$$a = a_1 + a_2 \quad \text{and} \quad a\, R\, b \quad (\text{or } b\, R\, a),$$

we have by 5.7

$$a = \sum_{i<\infty} d_i\,, \quad a_1 = d_0\,, \quad \text{and} \quad a_2 = \sum_{i<\infty} d_{i+1}\,.$$

Hence, by 6.7(ii),

$$b = \sum_{i<\infty} b'_i \quad \text{where} \quad d_i\, R\, b'_i \ (\text{or } b'_i\, R\, d_i) \quad \text{for} \quad i = 0, 1, 2, \cdots .$$

We put

$$b_1 = b'_0 \quad \text{and} \quad b_2 = \sum_{i<\infty} b'_{i+1}\,;$$

and we obtain with the help of 5.1.I and 6.4(ii)

$$b = b_1 + b_2\,, \quad \text{and} \quad a_i\, R\, b_i \ (\text{or } b_i\, R\, a_i) \quad \text{for} \quad i = 1, 2.$$

Hence, by 6.7(i), the conclusion.

An example of an equivalence relation R which is infinitely additive and finitely refining in a G.C.A. $\mathfrak{A}$, without being infinitely refining, can be constructed rather easily; such relations, however, rarely occur in practice.

THEOREM 6.10 (FUNDAMENTAL THEOREM ON COSET ALGEBRAS). *If $\mathfrak{A} = \langle A, +, \sum \rangle$ is a C.A., or a G.C.A., and if R is an infinitely additive and finitely (or infinitely) refining equivalence relation in $\mathfrak{A}$, then $\mathfrak{A}/R$ is again a C.A. or a G.C.A.*

PROOF: In view of 6.9, we can restrict ourselves to the case when R is finitely refining. By 6.6 it suffices to show that $\mathfrak{A}/R$ satisfies 1.1.VI, VII, or 5.1.IV, V. To obtain 1.1.VII, or 5.1.V, consider two sequences of cosets a_i/R and b_i/R with

$$(1) \qquad a_n/R = b_n/R + a_{n+1}/R \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

Then, by 6.3, there are two sequences of elements a'_i and b'_i in A such that

$$a_n\, R\, a'_n\,, \quad b_n\, R\, b'_n\,, \quad b'_n + a'_{n+1} \ \varepsilon \ A,$$
$$\text{and} \quad a_n\, R\, (b'_n + a'_{n+1}) \quad \text{for} \quad n < \infty .$$

Thus, in particular,

$$(b_0' + a_1') \; R \; a_0 \; ;$$

and, by 6.7(i), we obtain

$$\bar{a}_0 = \bar{b}_0 + \bar{a}_1 \quad \text{where} \quad \bar{a}_0 = a_0 , \quad b_0' \, R \, \bar{b}_0 , \quad \text{and} \quad a_1' \, R \, \bar{a}_1 .$$

We now have

$$(b_1' + a_2') \; R \; \bar{a}_1 \; ;$$

and hence, again by 6.7(i),

$$\bar{a}_1 = \bar{b}_1 + \bar{a}_2 \quad \text{with} \quad b_1' \, R \, \bar{b}_1 \quad \text{and} \quad a_2' \, R \, \bar{a}_2 .$$

By continuing this procedure indefinitely, we arrive at two infinite sequences $\bar{a}_i$ and $\bar{b}_i$ with

$$\bar{a}_n = \bar{b}_n + \bar{a}_{n+1} , \quad a_n \, R \, a_n' \, R \, \bar{a}_n , \quad \text{and} \quad b_n \, R \, b_n' \, R \, \bar{b}_n \quad \text{for} \quad n < \infty .$$

Hence, by applying 1.1.VII, or 5.1.V, to the original algebra $\mathfrak{A}$, we obtain an element $c \; \varepsilon \; A$ for which

$$\bar{a}_n = c + \sum_{i<\infty} \bar{b}_{n+i} \; ;$$

and with the help of 6.3–6.5 we easily conclude that

(2) $$a_n/R = c/R + \sum_{i<\infty} (b_{n+i}/R) \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

Thus, (1) implies (2); in other words, 1.1.VII, or 5.1.V, holds in $\mathfrak{A}/R$. The proof of 1.1.VI (or 5.1.IV) for $\mathfrak{A}/R$ is much simpler; we apply, of course, 1.1.VI to $\mathfrak{A}$, in addition to 6.3–6.5 and 6.7(i).

As trivial examples of relations to which 6.10 can be applied in every G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$, we may mention the UNIVERSAL RELATION R_1 which holds between any two elements $x, y \; \varepsilon \; A$; and the IDENTITY RELATION R_2 which holds between x and y if, and only if, $x = y$. Both R_1 and R_2 are infinitely additive and infinitely refining equivalence relations in $\mathfrak{A}$. $\mathfrak{A}/R_1$ is always a C.A., and is isomorphic with every C.A. (and G.C.A.) containing one element only; while $\mathfrak{A}/R_2$ is isomorphic with the original algebra $\mathfrak{A}$. On the other hand, consider the relation R_3 which holds between x and y in A if, and only if, either $x = y = 0$ or $x \neq 0$ and $y \neq 0$. R_3 is an infinitely additive equivalence relation in $\mathfrak{A}$, but in general is not even finitely refining. Nevertheless, if $\mathfrak{A}$ is a C.A., then $\mathfrak{A}/R$ is also a C.A.; in

fact, if $\mathfrak{A}$ contains at least two different elements, then $\mathfrak{A}/R$ is isomorphic with every C.A. containing exactly two elements. However, most of the equivalence relations which generate interesting and important examples of C.A.'s prove to be finitely, and even infinitely, refining.

Given any algebras $\mathfrak{A}_i$ correlated with elements i of an arbitrary set I, we can construct from them a new algebra $\mathfrak{P}$ —the cardinal (or direct) product of the algebras $\mathfrak{A}_i$. Usually two variants of this notion are distinguished, which will be referred to here as the STRONG and the WEAK CARDINAL PRODUCT. In the case of C.A.'s, however, a third, in a certain sense intermediate, variant of the same notion proves the most useful; we shall call it simply the CARDINAL PRODUCT. Although we shall mostly be concerned with the latter notion, we define here for further references all three variants just mentioned.

The notion of a cardinal product applies to arbitrary algebras provided they have a zero element (and even this restriction is unnecessary in the case of the strong product).

DEFINITION 6.11. *Let an algebra* $\mathfrak{A}_i = \langle A_i, +_i, \sum_i \rangle$ *with a zero element* 0_i *(in the sense of 1.2) be correlated with every element* i *of a given set* I. *Let* F_0 *be the set of all functions* f *such that* $D(f) = I$, *and* $f(i) \in A_i$ *for every* $i \in I$; *and let* F_1, *or* F_2, *be the set of all functions* $f \in F_0$ *for which the set of elements* i *with* $f(i) \neq 0_i$ *is finite, or at most denumerable.*

(i) *By the* SUM $f + g$ OF TWO FUNCTIONS $f, g \in F_0$ *we understand the function* $h \in F_0$ *such that*

$$h(i) = f(i) +_i g(i) \quad \textit{for every} \quad i \in I$$

(provided such a function exists). Similarly we define the SUM $\sum_{j<\infty} f_j$ OF AN INFINITE SEQUENCE OF FUNCTIONS $f_j \in F_0$. *The algebra* $\mathfrak{P}_k$ *constituted by the set* F_k, $k = 0, 1, 2$, *and the operations* $+$ *and* $\sum$ *thus defined is called the* STRONG, *the* WEAK, *or simply the* CARDINAL *(or* DIRECT*)* PRODUCT OF THE ALGEBRAS $\mathfrak{A}_i$, *in symbols,*

$$\mathfrak{P}_0 = \prod_{i \in I}^{s} \mathfrak{A}_i, \quad \mathfrak{P}_1 = \prod_{i \in I}^{w} \mathfrak{A}_i, \quad \textit{and} \quad \mathfrak{P}_2 = \prod_{i \in I} \mathfrak{A}_i.$$

(ii) *If the set* I *consists of two numbers,* 0 *and* 1, *and if* $\mathfrak{A}_0 = \mathfrak{A}$ *and* $\mathfrak{A}_1 = \mathfrak{B}$, *we put*

$$\mathfrak{P}_0 = \mathfrak{A} \times \mathfrak{B}.$$

(iii) *In case all algebras* $\mathfrak{A}_i$ *with* $i \in I$ *are identical with a given algebra* $\mathfrak{A}$, *we call* $\mathfrak{P}_k$ *the* STRONG, *the* WEAK, *or simply the* CARDINAL POWER OF $\mathfrak{A}$ WITH THE EXPONENT I, *in symbols,*

$$\mathfrak{P}_0 = \mathfrak{A}_s^I, \quad \mathfrak{P}_1 = \mathfrak{A}_w^I, \quad \text{and} \quad \mathfrak{P}_2 = \mathfrak{A}^I.$$

In case the set I is at most denumerable, the cardinal product is obviously identical with the strong cardinal product; and if I is finite, all three products coincide. This explains why we use only one symbol for the product of two algebras. If I is the set of all numbers $i < n \leqq \infty$, the functions of F_0 can be identified with n-termed sequences $\langle x_0, x_1, \cdots, x_i, \cdots \rangle$ with $x_i \in A_i$ for $i < n$. If I is empty, the set F_0 contains only one element—the 'empty' function or sequence.

THEOREM 6.12. *Let an algebra* $\mathfrak{A}_i = \langle A_i, +_i, \sum_i \rangle$ *with a zero element* 0_i *be correlated with every element* i *of a given set* I.

(i) *In order that* $\prod_{i \in I} \mathfrak{A}_i$ *be a C.A., or a G.C.A., or a finitely closed G.C.A., it is necessary and sufficient that every algebra* $\mathfrak{A}_i$ *with* $i \in I$ *be a C.A., or a G.C.A., or a finitely closed G.C.A.*

(ii) *The same applies to* $\prod_{i \in I}^{s} \mathfrak{A}_i$ *and—in the case of G.C.A.'s, or of finitely closed G.C.A.'s (but not of C.A.'s)—also to* $\prod_{i \in I}^{w} \mathfrak{A}_i$. *However,* $\prod_{i \in I}^{w} \mathfrak{A}_i$ *never satisfies* 1.1.II, *and hence is never a C.A., unless there are only finitely many elements* $i \in I$ *for which* A_i *contains two or more distinct elements.*

PROOF: by 1.1, 5.1, 5.26, and 6.11, without difficulty.

The last part of 6.12 makes it clear why the weak product plays a rather restricted part in the study of C.A.'s. On the other hand, as was mentioned before, the strong product coincides with the plain product in the important case when the set I is at most denumerable. Moreover, there are some general theorems concerning the cardinal product of C.A.'s which fail when applied to the strong product; and but very few more or less interesting facts are known which would apply specifically to strong products. Thus, the strong product appears to be a less appropriate instrument in dealing with C.A.'s than the plain product.

DEFINITION 6.13. *By a* SUBALGEBRA OF AN ALGEBRA $\mathfrak{A} = \langle A, +, \sum \rangle$ *we understand an algebraic system* $\mathfrak{B}$ *constituted by an arbitrary non-empty subset* B *of* A, *and by the fundamental operations* $+$ *and*

$\sum$ *of* $\mathfrak{A}$. *If both* $\mathfrak{A}$ *and* $\mathfrak{B}$ *belong to a class* K *of algebras,* $\mathfrak{B}$ *is called a* K-SUBALGEBRA *(e.g., a* CARDINAL, *or* GENERALIZED CARDINAL, SUBALGEBRA*) of* $\mathfrak{A}$. *Under these conditions we also say that the set* B *itself is a* SUBALGEBRA, *or a* K-SUBALGEBRA, OF $\mathfrak{A}$ *(or* OF A UNDER $+$ AND $\sum$*); and we call two subsets* B *and* B' *of* A ISOMORPHIC,

$$B \cong B',$$

if the corresponding subalgebras $\mathfrak{B}$ *and* $\mathfrak{B}'$ *are isomorphic.*

To give an example: the cardinal product of C.A.'s $\mathfrak{A}_i$, as defined in 6.11, is a cardinal subalgebra of the strong product of these algebras, and contains their weak product as a generalized cardinal subalgebra.

Notice that Definition 6.13 does not imply that a generalized cardinal subalgebra B of a G.C.A. A must be (relatively) closed under the operations $+$ and $\sum$; i.e., the sum of two or infinitely many elements of B may not be in B, even if this sum exists and is in A. Conversely, a subset B of A which is in this sense closed under both operations of addition is not necessarily a generalized cardinal subalgebra of A. On the other hand, a set B which is a cardinal subalgebra of a C.A. A must obviously be closed under $+$ and $\sum$. The converse, however, does not hold in this case either, in view of the existential character of some of the postulates occurring in the definition of C.A.'s.

Various fundamental problems concerning subalgebras remain open; for instance, the problem of the existence and of the structure of the smallest cardinal subalgebra which includes a given set. The general notion of a cardinal, or generalized cardinal, subalgebra will play no major part in our discussion; we introduce it mainly to simplify the formulation of certain theorems. In §9 we shall discuss, however, certain important particular cases of this notion—in fact, the notions of a semi-ideal and of an ideal.

§7. CLOSURES OF GENERALIZED CARDINAL ALGEBRAS

The following notion is of fundamental importance for the study of G.C.A.'s:

DEFINITION 7.1. *An algebra* $\bar{\mathfrak{A}} = \langle \bar{A}, \bar{+}, \bar{\sum} \rangle$ *is called a* CLOSURE OF AN ALGEBRA $\mathfrak{A} = \langle A, +, \sum \rangle$ *if the following conditions are satisfied:*

(i) $\mathfrak{A}$ *is a G.C.A.,* $\bar{\mathfrak{A}}$ *is a C.A., and* A *is a subset of* $\bar{A}$;

(ii) *for any elements* $a, a_0, a_1, \cdots, a_i, \cdots \varepsilon A$, *the formulas*

$$a = \sum_{i<\infty} a_i \quad \textit{and} \quad a = \bar{\sum_{i<\infty}} a_i$$

are equivalent;

(iii) *for every element* $a \,\varepsilon\, \bar{A}$ *there are elements* $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$ *such that*

$$a = \bar{\sum_{i<\infty}} a_i .$$

THEOREM 7.2. *If* $\bar{\mathfrak{A}} = \langle \bar{A}, \bar{+}, \bar{\sum} \rangle$ *is a closure of* $\mathfrak{A} = \langle A, +, \sum \rangle$, *then* $\mathfrak{A}$ *and* $\bar{\mathfrak{A}}$ *have the same zero element* $0 = \bar{0}$.

PROOF: From 1.34 and 7.1(iii) we conclude that the zero element $\bar{0}$ of $\bar{\mathfrak{A}}$ is in A. By putting

$$a_0 = \bar{0} \quad \text{and} \quad a_{i+1} = 0 \quad \text{for} \quad i = 0, 1, 2, \cdots$$

we obtain with the help of 5.16

$$\sum_{i<\infty} a_i = \bar{0}.$$

Hence, by 7.1(ii) and 1.34, the conclusion.

THEOREM 7.3. *Let* $\bar{\mathfrak{A}} = \langle \bar{A}, \bar{+}, \bar{\sum} \rangle$ *be a closure of* $\mathfrak{A} = \langle A, +, \sum \rangle$. *Then, for every integer* $n \leqq \infty$ *and for any elements* $a, b, c, a_0, a_1, \cdots, a_i, \cdots$ *in* A, *the formulas*

(i) $$a = \sum_{i<n} a_i \quad \textit{and} \quad a = \bar{\sum_{i<n}} a_i$$

are equivalent; and so are the formulas

(ii) $$a = b + c \quad \textit{and} \quad a = b \,\bar{+}\, c,$$

as well as

(ii) $$a = n \cdot b \quad \textit{and} \quad a = n \,\bar{\cdot}\, b.$$

PROOF: by 7.1 and 7.2, and with the help of 5.2 and 5.6.

THEOREM 7.4. *Let* $\bar{\mathfrak{A}} = \langle \bar{A}, \mp, \sum \rangle$ *be a closure of* $\mathfrak{A} = \langle A, +, \sum \rangle$. *Then, for every* $a \,\varepsilon\, A$, *the following two conditions are equivalent:*

(i) $$b \,\varepsilon\, A \text{ and } b \leqq a;$$
(ii) $$b \,\varepsilon\, \bar{A} \text{ and } b \mathrel{\bar{\leqq}} a.$$

PROOF: (i) obviously implies (ii) by 1.5, 5.2, 7.1(i), and 7.3(ii). Now assume (ii) to hold. We have then for some $c \,\varepsilon\, \bar{A}$

(1) $$b \mp c = a;$$

and hence, by 7.1(iii) and 1.1.IV,

(2) $$b = \sum_{i<\infty} b_i, \quad c = \sum_{i<\infty} c_i, \quad \text{and} \quad a = \sum_{i<\infty} b_i \mp \sum_{i<\infty} c_i = \sum_{i<\infty} (b_i \mp c_i)$$

where b_i and c_i are in A. By putting

$$a_{2\cdot i} = b_i \quad \text{and} \quad a_{2\cdot i+1} = c_i,$$

we obtain by 1.39 and 7.1(ii)

$$a = \sum_{i<\infty} a_i = \sum_{i<\infty} a_i .$$

Hence, by 5.11,

$$a_{2\cdot i} + a_{2\cdot i+1} = b_i + c_i \,\varepsilon\, A;$$

and therefore, by (2) and 7.3,

$$a = \sum_{i<\infty} (b_i + c_i).$$

By applying 5.1.II to the latter formula and then using 7.3 again, we conclude by means of (1) and (2) that

$$b,\ c \,\varepsilon\, A \quad \text{and} \quad b + c = a.$$

Hence (i) follows by 5.2 (1.5), and the proof is complete.

THEOREM 7.5. *Let* $\bar{\mathfrak{A}} = \langle \bar{A}, \mp, \sum \rangle$ *be a closure of* $\mathfrak{A} = \langle A, +, \sum \rangle$. *Then, for every integer* $n \leqq \infty$ *and for any* $a, a_0, a_1, \cdots, a_i, \cdots \varepsilon\, A$, *the formulas*

(i) $$a = \bigcup_{i<n} a_i \quad \text{and} \quad a = \bar{\bigcup_{i<n}} a_i$$

are equivalent; and if $n \neq 0$, so are the formulas

(ii) $$a = \bigcap_{i<n} a_i \quad \text{and} \quad a = \overline{\bigcap_{i<n}} a_i.$$

PROOF: by 3.16, 3.17, 5.2 (3.1, 3.2), 7.1(i), and 7.4. It is essential here that in case (i), when dealing with $\bar{\mathfrak{A}}$, we apply 3.17 and not the usual definition 3.2; it is irrelevant, however, whether we use 3.1 or 3.16 in case (ii).

THEOREM 7.6. *Let $\bar{\mathfrak{A}} = \langle \bar{A}, \bar{+}, \bar{\sum} \rangle$ be a closure of $\mathfrak{A} = \langle A, +, \sum \rangle$, and let $a \in A$. If a is idem-multiple, or finite, or multiple-free, or indecomposable in $\mathfrak{A}$, it has also the same property in $\bar{\mathfrak{A}}$; and conversely.*

PROOF: by 4.1, 4.10, 4.26, 4.38, 5.2, and 7.1–7.4.

In view of 7.1–7.6, we can use the same symbol and terms for arithmetical operations, relations, distinguished elements and classes of elements in a G.C.A. $\mathfrak{A}$ and its closure $\bar{\mathfrak{A}}$, without any danger of confusion. In particular, we can always assume without loss of generality that $\mathfrak{A}$ is a subalgebra of $\bar{\mathfrak{A}}$—in case both algebras are given in advance.

LEMMA 7.7. *Let I be the set of all integers $i < \infty$. Let $\mathfrak{A} = \langle A, +, \sum \rangle$ be a G.C.A., and let R be the relation which holds between two infinite sequences of elements x_i and y_j in A if, and only if, there is a double sequence of elements $r_{i,j}$ in A such that*

(i) $$x_i = \sum_{j<\infty} r_{i,j} \quad \text{and} \quad y_j = \sum_{i<\infty} r_{i,j} \quad \text{for any} \quad i < \infty \quad \text{and} \quad j < \infty.$$

Then $\mathfrak{A}^I/R$ is a C.A., and it is a closure of one of its subalgebras $\mathfrak{B}$ which is isomorphic with $\mathfrak{A}$.

PROOF: The algebra $\mathfrak{A}^I$ is a G.C.A. by 6.11 and 6.12. Its elements can be identified with infinite sequences

$$x = \langle x_0, x_1, \cdots, x_i, \cdots \rangle \text{ with } x_i \in A \text{ for } i = 0, 1, 2, \cdots;$$

let S be the set of all these sequences. The relation R is clearly reflexive; for, given $x \in S$, we put

$$r_{i,i} = x_i \quad \text{and} \quad r_{i,j} = 0 \quad \text{for} \quad i \neq j, \quad i < \infty, \quad j < \infty,$$

and we conclude from 5.16 that the double sequence thus defined satisfies the formulas (i) of our theorem (with $y_j = x_j$). R is ob-

viously symmetric. To show that it is transitive, consider x, y, $z \,\varepsilon\, S$ with $x \,R\, y$ and $y \,R\, z$. We then have two double sequences $r_{i,j}$ and $s_{j,k}$ such that

(1) $x_i = \sum_{j<\infty} r_{i,j}$, $y_j = \sum_{i<\infty} r_{i,j} = \sum_{k<\infty} s_{j,k}$, and $z_k = \sum_{j<\infty} s_{j,k}$

for any $i < \infty$, $j < \infty$, and $k < \infty$. By applying 5.23 to the second of these formulas, we see that we can correlate with every $j < \infty$ a double sequence $t_{i,k}^{(j)}$ such that

(2) $r_{i,j} = \sum_{k<\infty} t_{i,k}^{(j)}$ and $s_{j,k} = \sum_{i<\infty} t_{i,k}^{(j)}$ for $i < \infty$ and $k < \infty$.

(1) and (2) give

(3) $$x_i = \sum_{j<\infty} \sum_{k<\infty} t_{i,k}^{(j)} \quad \text{and} \quad z_k = \sum_{j<\infty} \sum_{i<\infty} t_{i,k}^{(j)}.$$

Consequently, by 5.22, we have

(4) $$u_{i,k} = \sum_{j<\infty} t_{i,k}^{(j)} \,\varepsilon\, A;$$

and from (3) and (4) we obtain, again with the help of 5.22,

$$x_i = \sum_{k<\infty} u_{i,k} \quad \text{and} \quad z_k = \sum_{i<\infty} u_{i,k} \quad \text{for} \quad i < \infty \quad \text{and} \quad k < \infty.$$

Hence, by (i), $x \,R\, z$, and R is transitive. In a similar (though simpler) way we show that R is infinitely additive and finitely refining in $\mathfrak{A}^I$; the proof of the first property is based upon 5.22, 6.4, and 6.11, while that of the second rests upon 5.1.II, 5.1.IV, 6.7, and 6.11. By applying 6.10 now, we arrive at the conclusion that

(5) $$\mathfrak{A}^I/R \text{ is a G.C.A.}$$

Furthermore, given a sequence of sequences $x^{(0)}, x^{(1)}, \cdots, x^{(i)}, \cdots \,\varepsilon\, S$, we define a new sequence of sequences $y^{(0)}, y^{(1)}, \cdots, y^{(i)}, \cdots \,\varepsilon\, S$ by putting, for any given $i < \infty$ and $j < \infty$,

(6) $y_{2^i\cdot(2\cdot j+1)}^{(i)} = x_j^{(i)}$, and $y_k^{(i)} = 0$ for every $k < \infty$ which is different from all numbers of the form $2^i\cdot(2\cdot l + 1)$, $l = 0, 1, 2, \cdots$.

From (6) it follows that, for any given $k < \infty$, there is at most one $i < \infty$ with

$$y_k^{(i)} \neq 0.$$

Hence, by 5.16,

$$\sum_{i<\infty} y_k^{(i)} \varepsilon A \quad \text{for} \quad k = 0, 1, 2, \cdots,$$

and consequently, by 6.11,

$$\sum_{i<\infty} y^{(i)} \varepsilon S. \tag{7}$$

Moreover, it is easily seen that

$$x^{(i)} R y^{(i)} \quad \text{for every} \quad i < \infty; \tag{8}$$

for, by putting

$$r_{j,k}^{(i)} = x_j^{(i)} \quad \text{for} \quad k = 2^i \cdot (2 \cdot j + 1), \quad \text{and} \quad r_{j,k}^{(i)} = 0 \quad \text{otherwise},$$

we conclude by 5.16 and (6) that

$$x_j^{(i)} = \sum_{k<\infty} r_{j,k}^{(i)} \quad \text{and} \quad y_k^{(i)} = \sum_{j<\infty} r_{j,k}^{(i)}$$

$$\text{for} \quad i < \infty, \quad j < \infty, \quad \text{and} \quad k < \infty.$$

By 6.3 and 6.4 (and in view of the properties of R previously established), formulas (7) and (8) imply that the sum of cosets

$$\sum_{i<\infty} (x^{(i)}/R)$$

exists. Hence, the G.C.A. $\mathfrak{A}^I/R$ satisfies 1.1.II; and therefore, by 5.24 and (5),

$$\mathfrak{A}^I/R \text{ is a C.A.} \tag{9}$$

We now correlate with every $a \varepsilon A$ the sequence $\bar{a} \varepsilon S$ defined as follows:

$$(\bar{a})_0 = a \quad \text{and} \quad (\bar{a})_{i+1} = 0 \quad \text{for} \quad i = 0, 1, 2, \cdots. \tag{10}$$

Let T be the set of all these sequences, and B the family of corresponding cosets; we put

$$\mathfrak{B} = \langle \mathsf{B}, +, \sum \rangle. \tag{11}$$

From (i), (10), 5.16, 5.20, and 6.3 we easily conclude that

(12) for any $a \varepsilon A$ and $x \varepsilon S$, the formulas

$$\bar{a}/R = x/R \quad \text{and} \quad a = \sum_{i<\infty} x_i$$

are equivalent.

Hence, by (10) and 5.16,

(13) for any $a, b \in A$, the formulas $\bar{a}/R = \bar{b}/R$ and $a = b$ are equivalent.

Furthermore, we obtain by means of (10), (12), 5.1.II, 5.4, 6.3–6.5, and 6.11 (and by applying various properties of R previously established):

(14) for any $a, b, c \in A$, the formulas $\bar{a}/R = \bar{b}/R + \bar{c}/R$ and $a = b + c$ are equivalent.

Similarly, by using 5.22 instead of 5.1.II,

(15) for any $a, a_0, a_1, \cdots, a_i, \cdots \in A$, the formulas

$$\bar{a}/R = \sum_{i<\infty} (\overline{a_i}/R) \quad \text{and} \quad a = \sum_{i<\infty} a_i$$

are equivalent.

From (11), (13)–(15) and 6.1 it is seen that the function

$$f(a) = \bar{a}/R \quad \text{for every} \quad a \in A$$

maps $\mathfrak{A}$ isomorphically onto $\mathfrak{B}$, and that consequently

(16) $$\mathfrak{A} \cong \mathfrak{B}.$$

Hence, by the hypothesis,

(17) $$\mathfrak{B} \text{ is a G.C.A.}$$

Finally, given an arbitrary sequence $x \in S$, we define a sequence of sequences $x^{(0)}, x^{(1)}, \cdots, x^{(i)}, \cdots \in S$ with

$$x_i^{(i)} = x_i, \quad \text{and} \quad x_j^{(i)} = 0 \quad \text{for} \quad i \neq j.$$

We then have by 5.16, 6.11, and (12)

$$x = \sum_{i<\infty} x^{(i)} \quad \text{and} \quad \overline{x_i}/R = x^{(i)}/R;$$

and hence, with the help of 6.3 and 6.4, we obtain

(18) $x/R = \sum_{i<\infty} (\overline{x_i}/R)$ where $\overline{x_i}/R \in \mathsf{B}$ for $i = 0, 1, 2, \cdots$.

By (9), (11), and (17), the algebras $\mathfrak{B}$ and $\mathfrak{A}^I/R$ satisfy condition 7.1(i); since, by (11) and 6.13, $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}^I/R$, condition 7.1(ii) is automatically satisfied; and, in view of (18), 7.1(iii) also holds. Hence

(19) $\mathfrak{A}^I/R$ is a closure of its subalgebra $\mathfrak{B}$.

By (9), (16), and (19), the proof is complete.

THEOREM 7.8 (IMBEDDING THEOREM). *For every G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$ *there exists a C.A.* $\bar{\mathfrak{A}}$ *which is a closure of* $\mathfrak{A}$.

PROOF: 7.8 can be derived from 7.7 by means of a procedure which is familiar from general algebra and does not involve any specific properties of C.A.'s.[1] Using the notation of 7.7, we first construct a C.A.

$$\mathfrak{A}' = \langle A', +', \textstyle\sum' \rangle \cong \mathfrak{A}^I/R$$

in which the set A' has no elements in common with A. By 7.7, $\mathfrak{A}'$ is a closure of one of its subalgebras

$$\mathfrak{B}' = \langle B', +', \textstyle\sum' \rangle \cong \mathfrak{A}.$$

By 'exchanging' now in $\mathfrak{A}'$ the elements of B' with the correlated elements of A, we arrive at an algebra $\bar{\mathfrak{A}}$ that satisfies the conclusion.

The imbedding theorem has many important implications. First of all, we have acquired in this theorem a universal method which enables us to extend automatically to G.C.A.'s all the results obtained in the arithmetic of C.A.'s, after having provided these results with appropriate existential assumptions. In fact, to establish any such result for a given G.C.A. $\mathfrak{A}$, we imbed $\mathfrak{A}$ in its closure $\bar{\mathfrak{A}}$. Since the result in question is assumed to hold in every C.A., it applies in particular to $\bar{\mathfrak{A}}$, and hence it applies to $\mathfrak{A}$ also, by virtue of 7.1–7.6.

Moreover, by means of the same method we can obtain new theorems of the arithmetic of G.C.A.'s whose direct derivation from the results previously established would present considerable difficulties. To illustrate the situation we have in mind here, consider the following theorem which applies to elements of an arbitrary G.C.A.:

If $a = c_1 + c_2 = c_3 + c_4$ *and* $b = c_1 + c_3 = c_2 + c_4$, *then* $a = b$.

In the arithmetic of C.A.'s this theorem is an immediate corollary of 2.34, since the hypothesis implies at once $2 \cdot a = 2 \cdot b$. In the domain of G.C.A.'s such a direct derivation is impossible, for $2 \cdot a$ may not exist; however, an application of 7.8 leads quickly to the desired result.

[1] Cf., e.g., van der Waerden [1], p. 42.

With the help of 7.8 we can also establish various existential theorems which apply to arbitrary G.C.A.'s; they compensate to some extent for the lack of the general closure postulates, and permit us in turn to simplify and improve other results by removing superfluous existential assumptions. The method involved here can be described as follows. To prove the existence of an element (a sum, an upper bound, etc.) in a G.C.A. $\mathfrak{A}$, we again imbed $\mathfrak{A}$ in its closure $\bar{\mathfrak{A}}$. If now we succeed in showing that the element in question not only exists in $\bar{\mathfrak{A}}$ but is less than or equal to another element whose existence in $\mathfrak{A}$ has been assumed, we apply 7.4 and conclude that our element belongs to $\mathfrak{A}$. We give here a few results obtained by this method; they apply to elements of an arbitrary G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$.

THEOREM 7.9. *If* $n \leqq \infty$, $p \leqq \infty$,

$$\sum_{i<n} a_i \,\varepsilon\, A,$$

and if $k_0, k_1, \cdots, k_i, \cdots$ *is a p-termed sequence without repeating terms and with* $k_i < n$ *for* $i < p$, *then*

$$\sum_{i<p} a_{k_i} \,\varepsilon\, A \quad \textit{and} \quad \sum_{i<p} a_{k_i} \leqq \sum_{i<n} a_i .$$

PROOF: The inequality in the conclusion for C.A.'s follows from 1.40; and hence the theorem can be obtained by 7.3, 7.4, and 7.8.

THEOREM 7.10. *If*

$$\sum_{i<n} a_i \,\varepsilon\, A \quad \textit{and} \quad \sum_{i<n} a_i \leqq b \quad \textit{for every } n < \infty,$$

then

$$\sum_{i<\infty} a_i \,\varepsilon\, A \quad \textit{and} \quad \sum_{i<\infty} a_i = \bigcup_{n<\infty} \sum_{i<n} a_i \leqq b.$$

PROOF: by 2.21, 3.19, 7.3–7.5, and 7.8.

THEOREM 7.11. *Let* n *be* $\leqq \infty$. *If* $a_i \leqq a_{i+1} \leqq b$ *for every* i *with* $i + 1 < n$; *or, more generally, if*

$$\bigcap_{i<p} a_{k_i} \,\varepsilon\, A \quad \textit{and} \quad \bigcap_{i<p} a_{k_i} \leqq b$$

for every p-termed sequence $a_{k_0}, a_{k_1}, \cdots, a_{k_i}, \cdots$ *with* $0 < p < \infty$, *and* $k_i < n$ *for* $i < p$; *then*

$$\bigcup_{i<n} a_i \,\varepsilon\, A \quad \textit{and} \quad \bigcup_{i<n} a_i \leqq b.$$

PROOF: by 3.34, 7.4, 7.5, and 7.8.

THEOREM 7.12. *Let n be* $\leqq \infty$; *and assume that either*

(i) $$a_i \cap a_j = 0 \text{ for all } i \text{ and } j \text{ with } i < j < n,$$

or else

(ii) *all the elements* a_i *with* $i < n$ *are idem-multiple.*

If there is an element b with $a_i \leqq b$ *for every* $i < n$ (*thus, in particular, if* $\sum_{i<n} a_i$ *or* $\bigcup_{i<n} a_i$ *is in* A), *then*

$$\sum_{i<n} a_i \,\varepsilon\, A \quad \text{and} \quad \sum_{i<n} a_i = \bigcup_{i<n} a_i .$$

Moreover, in the case (ii), $\sum_{i<n} a_i$ *is idem-multiple as well.*

PROOF: by 3.23, 4.7, 7.3–7.6, and 7.8.

From now on we shall freely apply to G.C.A.'s all the arithmetical theorems of Part I, provided with only those existential assumptions which cannot be dispensed with in the way discussed above. When using elementary theorems of §1 or §5, we shall not refer to them explicitly.

In particular, theorems of Part I can now be applied to coset algebras $\mathfrak{A}/R$ of G.C.A.'s $\mathfrak{A}$, in case these coset algebras prove to be G.C.A.'s themselves. The results thus obtained can always be put in a form in which they involve no cosets, but only elements of a given algebra and the relation R between these elements. We arrive in this way at certain generalizations of our arithmetical theorems—generalizations which, roughly speaking, consist in replacing the identity relation by a variable relation R subjected to certain conditions. It may be worth while formulating explicitly some of the results which we have in mind:

THEOREM 7.13. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A., and let R be an infinitely additive and finitely* (*or infinitely*) *refining equivalence relation in* $\mathfrak{A}$. *We then have for any elements* $a, a', a_0, a_1, \cdots, b, b', b_0, b_1, \cdots, c, c', c_0, c_1, \cdots$ *in* A (*assuming that the sums involved are also in* A):

(i) *if* $a \leqq b$, $a' \leqq b'$, $a\,R\,b'$, *and* $b\,R\,a'$, *then* $a\,R\,a'$ *and* $b\,R\,b'$;

(ii) *if* $a \leqq b \leqq c$ *and* $a\,R\,c$, *then* $a\,R\,b\,R\,c$;

(iii) *if* $n \leqq \infty$, *then*

$$(\sum_{i<n} a_i + b)\,R\,b$$

implies that $(a_i + b)\,R\,b$ *for every* $i < n$, *and conversely;*

(iv) *if* $c_0 \, R \, c_i$ *for* $i = 1, 2, 3$, *and* $(a + c_0 + c_2) \, R \, (b + c_1 + c_3)$, *then* $(a + c_0) \, R \, (b + c_1)$;

(v) *if* $0 < m < \infty$, $a_0 \, R \, a_i$ *and* $b_0 \, R \, b_i$ *for every* $i < m$, *and*

$$(\sum_{i<m} a_i) \, R \, (\sum_{i<m} b_i),$$

then $a_0 \, R \, b_0$;

(vi) *if* $a \leqq b \leqq c$, $a' \leqq c'$, $a \, R \, a'$, *and* $c \, R \, c'$, *then there is an element* $b' \, \varepsilon \, A$ *with* $a' \leqq b' \leqq c'$ *and* $b \, R \, b'$;

(vii) *if, for every* $n < \infty$, *there is an element* $d_n \, \varepsilon \, A$ *with*

$$(\sum_{i<n} a_i) \, R \, d_n \leqq b,$$

then there is an element $d \, \varepsilon \, A$ *with*

$$(\sum_{i<\infty} a_i) \, R \, d \leqq b.$$

Conclusions (i)–(v) *hold, more generally, for every infinitely additive equivalence relation* R *in* $\mathfrak{A}$ *for which* $\mathfrak{A}/R$ *is a G.C.A.*

PROOF: (i) By passing to cosets, we have (with the help of 6.3–6.5)

$$b/R = a'/R \leqq b'/R = a/R \leqq b/R.$$

Hence, by applying 1.31 to $\mathfrak{A}/R$ (which is a G.C.A. by 6.10), we obtain

$$a/R = b/R;$$

and by 6.3 we arrive at the conclusion.

To obtain (ii) we put in (i)

$$a' = b \quad \text{and} \quad b' = c.$$

(iii)–(v) can be derived in the same way as (i) by means of 1.47, 2.12, and 2.34.

By the hypothesis of (vi), we have for some x

(1) $$a' + x = c'.$$

By passing to cosets, we obtain

$$a'/R = a/R \leqq b/R \leqq c/R = c'/R \quad \text{and} \quad a'/R + x/R = c'/R.$$

Hence

$$a/R \leqq b/R \leqq a/R + x/R;$$

and therefore by 2.27 (with $n = 1$), there is a coset y/R such that

(2) $$b/R = a/R + y/R \quad \text{and} \quad y/R \leqq x/R.$$

From the second part of (2) we conclude, with the help of 6.3 and 6.8, that there is an element $y' \varepsilon A$ with

(3) $$y/R = y'/R \quad \text{and} \quad y' \leqq x.$$

Thus, by (1)–(3),

$$a' \leqq a' + y' \leqq a' + x = c'$$

and

$$b/R = a'/R + y'/R, \quad \text{i.e.} \quad b \, R \, (a' + y').$$

This shows that the element $b' = a' + y'$ satisfies the conclusion of (vi).

Finally, (vii) can be derived from 2.21 in the same way as (vi) from 2.27.

Theorem 7.13 can also be obtained in a more direct way, without making any use of the properties of coset algebras. Some parts of the proof, however, become very involved then; this applies especially to the derivation of (v). We want to outline here a direct proof of 7.13(ii), which is relatively simple and instructive.

By hypothesis there are elements b_0 and c_0 for which

(1) $$b = a + b_0 \quad \text{and} \quad c = b + c_0 .$$

We put

(2) $$a_0 = a,$$

and we obtain with the help of 5.8, 5.9, (1), (2), and the hypothesis

$$a_0 \, R \, (b_0 + c_0 + a_0).$$

Since the relation R is assumed to be finitely refining, we can now proceed as in the proof of 6.10. By applying 6.7 repeatedly we construct three infinite sequences of elements a_i, b_i, and c_i such that, for every $n < \infty$,

(3) $$a_n = b_{n+1} + c_{n+1} + a_{n+1} ,$$

(4) $$a_{n+1} \, R \, a_n , \quad b_{n+1} \, R \, b_n , \quad \text{and} \quad c_{n+1} \, R \, c_n .$$

We now apply 5.1.V to (3), and we conclude with the help of 5.1.II and (2) that there is an element d for which

(5) $$a = d + \sum_{i<\infty} b_{i+1} + \sum_{i<\infty} c_{i+1}.$$

From (1) and (5) we easily obtain (by 5.3, 5.8, and 5.9)

(6) $$b = d + \sum_{i<\infty} b_i + \sum_{i<\infty} c_{i+1} \quad \text{and} \quad c = d + \sum_{i<\infty} b_i + \sum_{i<\infty} c_i.$$

The relation R being by hypothesis reflexive and infinitely additive, formulas (4)–(6) imply by 6.4:

$$a\,R\,b \quad \text{and} \quad b\,R\,c,$$

i.e., the conclusion of 7.13(ii).

Notice that the symmetry and the transitivity of R are not involved in this argument; thus, 7.13(ii) applies to every relation R which is infinitely additive, finitely refining, and reflexive in a G.C.A. $\mathfrak{A}$.

We return now to the discussion of C.A.'s and G.C.A.'s in their relation to general algebraic notions which were introduced in §6. We are going to establish here a few simple extension theorems, 7.14–7.17, which—when combined with the imbedding theorem—reduce to some extent the algebraic study of G.C.A.'s to that of C.A.'s. This is, of course, important since C.A.'s are much more 'perfect' algebraic systems than G.C.A.'s. It should be noticed, however, that the reduction is not complete, for the variety of G.C.A.'s is much larger than that of C.A.'s. As is seen from Theorem 7.14, the closure $\bar{\mathfrak{A}}$ of a G.C.A. $\mathfrak{A}$ is uniquely determined by $\mathfrak{A}$ (up to isomorphism); but the converse does not hold.

Theorem 7.14. *Let $\bar{\mathfrak{A}} = \langle \bar{A}, +, \sum \rangle$ be a closure of its subalgebra $\mathfrak{A} = \langle A, +, \sum \rangle$, and $\bar{\mathfrak{A}'} = \langle \bar{A'}, +', \sum' \rangle$ be a closure of its subalgebra $\mathfrak{A}' = \langle A', +', \sum' \rangle$. If the function f maps $\mathfrak{A}$ isomorphically onto $\mathfrak{A}'$, then there is one and only one function $\bar{f}$ which maps $\bar{\mathfrak{A}}$ isomorphically onto $\bar{\mathfrak{A}'}$ in such a way that $\bar{f}(x) = f(x)$ for every $x \in A$. Hence, if $\mathfrak{A} \cong \mathfrak{A}'$, then $\bar{\mathfrak{A}} \cong \bar{\mathfrak{A}'}$.*

Proof: For every a in $\bar{A}$ we have by 7.1(iii) (and 6.13)

$$a = \sum_{i<\infty} a_i \quad \text{with} \quad a_0, a_1, \cdots, a_i, \cdots \in A;$$

hence, by 6.1(ii) and 6.13, the element b with

$$b = \sum_{i<\infty}{}' f(a_i)$$

is in $\overline{A'}$. By making an essential use of 2.1 (with $n = p = \infty$), 7.4, and 1.44, we show that this element b does not depend on the way in which the elements a_i have been chosen; we can thus put

$$b = \bar{f}(a).$$

In an analogous way we show that, for every b in $\overline{A'}$, there exists an element $a \,\varepsilon\, \bar{A}$ with $b = \bar{f}(a)$. Thus, $\bar{f}$ satisfies 6.1(ii). The proof that it satisfies 6.1(iii) is quite elementary; and 6.2 gives the conclusion.

There is a striking analogy between 7.8 and 7.14 on the one hand, and certain familiar theorems of modern algebra on the other; we have in mind here theorems on the imbedding of a commutative semi-group in the group of differences, or of an integrity domain in the field of quotients, or various theorems on extensions of a field.[2] We note, however, one important difference. Every field which contains a given integrity domain also contains the field of quotients of this domain as a subfield. However, the fact that a G.C.A. $\mathfrak{B}$ is a subalgebra of a C.A. $\mathfrak{A}$ does not always imply that $\mathfrak{A}$ contains a cardinal subalgebra $\bar{\mathfrak{B}}$ which is a closure of $\mathfrak{B}$; a counter-example can readily be constructed in the C.A. of integers discussed in Part III, §14. It is easily seen, however, that the closure of a G.C.A. $\mathfrak{B}$ has a weaker 'minimal property': if $\mathfrak{B}$ is a subalgebra of a C.A. $\mathfrak{A}$ and if there is a subalgebra $\bar{\mathfrak{B}}$ of $\mathfrak{A}$ which is a closure of $\mathfrak{B}$, then every cardinal subalgebra $\mathfrak{B}'$ of $\mathfrak{A}$ which contains $\mathfrak{B}$ must also contain $\bar{\mathfrak{B}}$ as a subalgebra. Cf. in this connection also Theorem 9.23 below.

THEOREM 7.15. *Let $\bar{\mathfrak{A}} = \langle \bar{A}, +, \sum \rangle$ be a closure of its subalgebra $\mathfrak{A} = \langle A, +, \sum \rangle$. If R is an infinitely additive and infinitely refining equivalence relation in $\mathfrak{A}$, then there is one and only one infinitely additive and infinitely refining equivalence relation $\bar{R}$ in $\bar{\mathfrak{A}}$ which coincides with R when applied to elements of A. If, conversely, $\bar{R}$ is an infinitely additive and infinitely refining equivalence relation in $\bar{\mathfrak{A}}$, then the relation R obtained by restricting $\bar{R}$ to elements of A is an infinitely additive and infinitely refining equivalence relation in $\mathfrak{A}$.*

[2] Cf., e.g., Birkoff-MacLane [1], p. 48, or van der Waerden [1], pp. 42 ff. and 99 ff.

PROOF: Given R, we define $\bar{R}$ in the following way:
(1) $a \bar{R} b$ if $a, b \varepsilon \bar{A}$ and if there are elements $a_0, a_1, \cdots, a_i, \cdots, b_0, b_1, \cdots, b_i, \cdots \varepsilon A$ with

$$a = \sum_{i<\infty} a_i, \quad b = \sum_{i<\infty} b_i, \quad \text{and} \quad a_i R b_i \quad \text{for} \quad i = 0, 1, 2, \cdots.$$

By (1), 7.1, and the hypothesis, $\bar{R}$ is obviously reflexive in $\bar{A}$ and symmetric; it can easily be shown, with the help of 1.42, 6.4, and 7.2, to be infinitely additive, and to coincide with R when applied to elements of A. To show that it is transitive, consider elements $a, b, c \varepsilon \bar{A}$ with

$$a \bar{R} b \quad \text{and} \quad b \bar{R} c.$$

By (1) we have

(2) $$a = \sum_{i<\infty} a_i, \quad b = \sum_{i<\infty} b_i = \sum_{j<\infty} b'_j, \quad c = \sum_{j<\infty} c_j,$$

(3) $$a_i, b_i, b'_j, c_j \varepsilon A, \quad a_i R b_i, \quad \text{and} \quad b'_j R c_j \quad \text{for} \quad i < \infty \quad \text{and} \quad j < \infty.$$

By applying 2.1 to the second formula of (2), we obtain

(4) $$b_i = \sum_{j<\infty} b_{i,j} \quad \text{and} \quad b'_j = \sum_{i<\infty} b_{i,j} \quad \text{for} \quad i < \infty \quad \text{and} \quad j < \infty.$$

By (3), (4), and 7.4 all elements $b_{i,j}$ are in A; and since R is infinitely refining in $\mathfrak{A}$, we obtain, by 6.7, refinements of a_i and c_j:

(5) $$a_i = \sum_{j<\infty} a_{i,j}, \quad c_j = \sum_{i<\infty} c_{i,j}, \quad a_{i,j} R b_{i,j} R c_{i,j}$$

where the elements $a_{i,j}$ and $c_{i,j}$ are again in A. By (2) and (5), and in view of the transitivity of R, we have

$$a = \sum_{i<\infty} \sum_{j<\infty} a_{i,j}, \quad c = \sum_{i<\infty} \sum_{j<\infty} c_{i,j} \quad \text{and}$$

$$a_{i,j} R c_{i,j} \quad \text{for} \quad i, j = 0, 1, 2, \cdots;$$

and hence, by (1) and with the help of 1.42,

$$a \bar{R} c.$$

Thus, $\bar{R}$ is transitive. The proof that $\bar{R}$ is infinitely refining is similar, though somewhat simpler; 2.1, 6.7, and 7.4 are again applied, but 1.42 is not. The uniqueness of $\bar{R}$ is a consequence of 6.4 and 7.1.

The second part of the theorem follows obviously from 6.4, 6.7, and 7.4.

By means of a similar argument it can be shown that every infinitely refining equivalence relation R in a G.C.A. $\mathfrak{A}$ can be extended to an equivalence relation $\bar{R}$ which is both infinitely additive and infinitely refining in $\mathfrak{A}$.

THEOREM 7.16. *If $\bar{\mathfrak{A}} = \langle \bar{A}, +, \sum \rangle$ is a closure of its subalgebra $\mathfrak{A} = \langle A, +, \sum \rangle$, and if R and $\bar{R}$ are two relations satisfying the conditions of* 7.15, *then the C.A. $\bar{\mathfrak{A}}/\bar{R}$ is a closure of an algebra $\mathfrak{B}$ which is isomorphic with the G.C.A. $\mathfrak{A}/R$.*

PROOF: We put

$$\mathfrak{B} = \langle \mathsf{C}, +, \sum \rangle$$

where C is the family of all cosets $a/\bar{R}$ with $a \in A$, and where the operations $+$ and $\sum$ are defined as in 6.3. The proof that $\mathfrak{B}$ satisfies the conclusion is based upon 6.1, 6.4–6.9, 7.4, and 7.15, and is quite elementary.

THEOREM 7.17. *If a G.C.A. $\mathfrak{A}_i$ and its closure $\bar{\mathfrak{A}}_i$ are correlated with every element i of a given set I, then $\prod_{i \in I} \bar{\mathfrak{A}}_i$ is a closure of $\prod_{i \in I} \mathfrak{A}_i$.*

PROOF: by 6.11, 6.12, and 7.1.

It is easily seen that 7.17 holds neither for strong nor for weak cardinal products.

We conclude this section by defining the notion of a FINITE CLOSURE which is related to that of closure. We shall apply this notion to various kinds of algebra, and therefore we formulate its definition in a general way, without referring specifically to G.C.A.'s.

DEFINITION 7.18. *An algebra $\mathfrak{A}' = \langle A', +', \sum' \rangle$ is called a* FINITE CLOSURE OF AN ALGEBRA *$\mathfrak{A} = \langle A, +, \sum \rangle$ if the following conditions are satisfied:*

(i) *$\mathfrak{A}'$ is finitely closed, and A is a subset of A';*

(ii) *if $a, b, c \in A$, then the formulas*

$$a = b + c \quad \text{and} \quad a = b +' c$$

are equivalent;

(iii) *if $b, c \in A'$ and $b +' c \in A$, then $b, c \in A$;*

(iv) *for every element $a \varepsilon A'$ there are elements $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$ with $i < n$, $0 < n < \infty$, such that*

$$a = \sum_{i<n}' a_i.$$

Let us assume that the addition of finite sequences has been defined in $\mathfrak{A}$ and $\mathfrak{A}'$ recursively in terms of binary addition. (We use for this purpose the first two formulas of 1.8, as well as 1.17 for $n \neq 0$. Of course, $\sum_{i<0} a_i$ is assumed to exist only if $\mathfrak{A}$ has the zero element; and $\sum_{i<n+1} a_i$ for $n \neq 0$ is in A only if $\sum_{i<n} a_i$ and $\sum_{i<n} a_i + a_n$ are in A.) Under this assumption, the infinite addition $\sum$ is not involved in conditions 7.18(i)–(iii) at all. Therefore, the notion of a finite closure can be automatically applied to algebraic systems $\mathfrak{A} = \langle A, + \rangle$ with one fundamental operation; such systems will be discussed in the third part of the work.

If, furthermore, the operations $+$ and $+'$ satisfy the associative law 5.9, we can show that conditions 7.18(ii),(iii) can be equivalently replaced by the following:

for any elements $a, a_0, a_1, \cdots, a_i, \cdots \varepsilon A$ with $i < n < \infty$, the formulas

$$a = \sum_{i<n} a_i \quad \textit{and} \quad a = \sum_{i<n}' a_i$$

are equivalent.

Definition 7.18 thus transformed assumes a form entirely analogous to that of 7.1.

From 7.8 and 7.18 we easily conclude that for every G.C.A. $\mathfrak{A}$ there is a finitely closed G.C.A. $\mathfrak{A}'$ which is a finite closure of $\mathfrak{A}$; in fact, we first construct a closure $\overline{\mathfrak{A}}$ of $\mathfrak{A}$, and we then define $\mathfrak{A}'$ as the subalgebra of $\overline{\mathfrak{A}}$ constituted by those elements which are sums of finitely many elements of $\mathfrak{A}$. We can also extend 7.14 to finite closures; i.e., we can show that every G.C.A. $\mathfrak{A}$ determines up to isomorphism the G.C.A. $\mathfrak{A}'$ which is its finite closure. Finally, it is easily seen that Theorem 7.17 remains valid if we replace in it closures by finite closures, and cardinal products by weak cardinal products.

§8. IDEM-MULTIPLE AND MULTIPLE-FREE ALGEBRAS

We are going to discuss in this section two special kinds of algebras —IDEM-MULTIPLE and MULTIPLE-FREE G.C.A.'s—as well as certain simple methods of constructing these algebras from arbitrary G.C.A.'s. It will be seen in our further discussion that the method of constructing multiple-free G.C.A.'s (described below in 8.16 and 8.17), when combined with that of coset algebras, leads to many important examples of G.C.A.'s.

DEFINITION 8.1. *An algebra* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is called* IDEM-MULTIPLE *if every element* $a \,\varepsilon\, A$ *is idem-multiple, i.e., satisfies the formula* $a + a = a$.

Compare here Definition 4.1.

THEOREM 8.2. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A., and let* n *be a number with* $2 \leqq n \leqq \infty$. *For* $\mathfrak{A}$ *to be idem-multiple it is necessary and sufficient that* $\mathfrak{A}$ *satisfy the following condition: if the elements* $a_0, a_1, \cdots, a_i, \cdots$ *with* $i < n$ *and* b *are in* A, *and if* $a_i \leqq b$ *for every* $i < n$, *then*

$$\sum_{i<n} a_i \,\varepsilon\, A \quad \textit{and} \quad \sum_{i<n} a_i = \bigcup_{i<n} a_i.$$

PROOF: The necessity follows at once from 7.12 and 8.1. If, on the other hand, the condition of the theorem is satisfied, we apply it to the case when

$$a_0 = a_1 = \cdots = a_i = \cdots = b = a$$

where a is any element in A, and by 4.3 and 8.1 we conclude that $\mathfrak{A}$ is idem-multiple.

COROLLARY 8.3. *For an idem-multiple G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$ *to be a C.A. it is sufficient (and necessary) that, for any elements* $a_0, a_1, \cdots, a_i, \cdots \,\varepsilon\, A$ *with* $i < \infty$ *there exist an element* $b \,\varepsilon\, A$ *with* $a_i \leqq b$ *for* $i = 0, 1, 2, \cdots$.

PROOF: by 5.24 and 8.2 (or 5.25 and 8.1).

THEOREM 8.4. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a C.A., or a G.C.A., and if* B *is the set of all idem-multiple elements in* A, *then* $\mathfrak{B} = \langle B, +, \sum \rangle$ *is an idem-multiple cardinal, or generalized cardinal, subalgebra of* $\mathfrak{A}$.

PROOF: Postulates 1.1.I–V (or 5.1.I–III) hold in $\mathfrak{B}$ by 4.2 and 4.7. If any idem-multiple elements a, b, and c, in A satisfy the hypothesis of Postulate 1.1.VI (or 5.1.IV), and if the elements a_i and b_i satisfy its conclusion, we easily show that the idem-multiple elements $\infty \cdot a_i$ and $\infty \cdot b_i$ also satisfy this conclusion; we apply here 4.3 with $n = \infty$ and 4.5. In a similar way we derive 1.1.VII (or 5.1.V). Thus, $\mathfrak{B}$ is a C.A. (or G.C.A.); the conclusion follows by 6.13 and 8.1.

THEOREM 8.5. *Let $\mathfrak{A} = \langle A, +, \sum \rangle$ be a C.A.; and let R be the relation which holds between two elements a and b in A if, and only if, $\infty \cdot a = \infty \cdot b$. Then R is an infinitely additive equivalence relation in $\mathfrak{A}$; and the coset algebra $\mathfrak{A}/R$ is an idem-multiple C.A. which is isomorphic with the subalgebra $\mathfrak{B} = \langle B, +, \sum \rangle$ of* 8.4.

PROOF: R is clearly an equivalence relation which is infinitely additive in $\mathfrak{A}$ (by 6.4). From 1.42, 4.3, 4.5, and 6.3 we easily see that the function $f(a) = a/R$ maps B onto the family C of all cosets in a one-to-one way; f also satisfies the conditions (iii) and (iv) of 6.1. Thus,

$$\mathfrak{B} \cong \mathfrak{A}/R;$$

and, in view of 8.4, the proof is complete.

Notice that the relation R of 8.5 is not necessarily a refining relation.

THEOREM 8.6. *If $\mathfrak{A}$ is an idem-multiple G.C.A. and $\bar{\mathfrak{A}}$ is a closure of $\mathfrak{A}$, then $\bar{\mathfrak{A}}$ is an idem-multiple C.A.*

PROOF: by 4.7, 7.1, and 8.1.

It can easily be shown that all general methods of construction discussed in §6, when applied to idem-multiple algebras, yield new idem-multiple algebras.

DEFINITION 8.7. *For every algebra $\mathfrak{A} = \langle A, +, \sum \rangle$ we put*

$$\breve{\mathfrak{A}} = \langle A, \cup, \bigcup \rangle$$

(where '$\cup$' and '$\bigcup$' denote the operations of forming the least upper bound defined in 3.2).

In connection with this definition it should be pointed out that the operation denoted by the symbol '$\bigcup$' applies to arbitrary systems of elements a_i correlated with elements i of any set I. When using

this symbol in the denotation of an algebra, however, we shall understand that the operation involved has been restricted to infinite sequences. An analogous remark applies to some other operation symbols which will occur in our further discussion.

COROLLARY 8.8. *If* $\mathfrak{A}$ *is a G.C.A., then* $\breve{\mathfrak{A}}$ *is an idem-multiple algebra in which the zero element and the relation* $\leqq$ *coincide with those in* $\mathfrak{A}$.

PROOF: by 3.2, 8.1, and 8.7.

COROLLARY 8.9. *For a G.C.A.* $\mathfrak{A}$ *to be idem-multiple it is necessary and sufficient that* $\mathfrak{A} = \breve{\mathfrak{A}}$.

PROOF: by 8.2 and 8.8.

The fact that an algebra $\mathfrak{A}$ is a G.C.A. does not imply, in general, that $\breve{\mathfrak{A}}$ is also a G.C.A. We have in this connection:

THEOREM 8.10. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A. with the following property:*

(i) *if* $a_0, a_1, \cdots, a_i, \cdots, b \in A$ *and* $a_i \leqq b$ *for every* $i < \infty$, *then* $\bigcap_{i<\infty} a_i$ *exists.*

Then $\breve{\mathfrak{A}}$ *is an idem-multiple G.C.A.*

PROOF: From (i) and 7.11 we conclude that $\bigcup_{i<n} a_i$ exists for every n-termed sequence $a_0, a_1, \cdots, a_i, \cdots, i < n \leqq \infty$, which is bounded above by a certain element b. Hence, as is easily seen, Postulates 5.1.I, II hold in $\breve{\mathfrak{A}}$; the same obviously applies to 5.1.III. The derivation of 5.1.IV is simple: given

$$a \cup b = \bigcup_{i<\infty} c_i \in A,$$

we notice that the elements

$$a_i = a \cap c_i \quad \text{and} \quad b_i = b \cap c_i \quad \text{for} \quad i = 0, 1, 2, \cdots$$

exist by (i) and satisfy the conclusion of 5.1.IV by 3.32. To obtain 5.1.V assume that

$$(1) \qquad a_n = b_n \cup a_{n+1} \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

The sequence $a_0, a_1, \cdots, a_i, \cdots$ being decreasing, the element

$$(2) \qquad c = \bigcap_{i<\infty} a_i = \bigcap_{i<\infty} a_{n+i} \quad \text{for} \quad n = 0, 1, 2, \cdots$$

exists by (i); and from what was said before it follows that $\bigcup_{j<\infty} b_{n+j}$ also exists for every $n < \infty$. From (1) it is easily seen that

$$a_n = a_{n+p} \cup \bigcup_{j<\infty} b_{n+j} \quad \text{for} \quad n < \infty \quad \text{and} \quad p < \infty.$$

Hence, by 3.30,

$$(3) \qquad a_n = \bigcap_{i<\infty} (a_{n+i} \cup \bigcup_{j<\infty} b_{n+j}) = \bigcap_{i<\infty} a_{n+i} \cup \bigcup_{j<\infty} b_{n+j}.$$

By (2) and (3) we have

$$a_n = c \cup \bigcup_{j<\infty} b_{n+j} \quad \text{for} \quad n = 0, 1, 2, \cdots;$$

i.e., c satisfies the conclusion of 5.1.V. Hence, and in view of 8.8, $\mathfrak{A}$ is an idem-multiple G.C.A.

Definition 8.11. *An algebra $\mathfrak{A} = \langle A, +, \sum \rangle$ is called* MULTIPLE-FREE *if, a being any element in A, $a + a$ is never in A, unless a is the zero element of $\mathfrak{A}$.*

Corollary 8.12. *A C.A. or, more generally, a finitely closed algebra $\mathfrak{A} = \langle A, +, \sum \rangle$ is never multiple-free, unless A contains no element $a \neq 0$.*

Proof: by 5.26 and 8.11.

Theorem 8.13. *Let $\mathfrak{A} = \langle A, +, \sum \rangle$ be a G.C.A., and let n be an integer with $2 \leqq n \leqq \infty$. Then the following conditions are equivalent:*

(i) *$\mathfrak{A}$ is multiple-free;*

(ii) *every element in A is multiple-free;*

(iii) *if the elements $a_0, a_1, \cdots, a_i, \cdots$ and $\sum_{i<n} a_i$ are in A, then $a_i \cap a_j = 0$ for all i and j with $i < j < n$.*

Proof: (i) implies (ii) by 4.26 and 8.11; (ii) implies (iii) by 4.32. By applying (iii) to the sequence $a_0, a_1, \cdots, a_i, \cdots$ with

$$a_0 = a_1 = a \quad \text{and} \quad a_i = 0 \quad \text{for} \quad 2 \leqq i < n,$$

and with the help of 8.11, we obtain (i). Hence, conditions (i)–(iii) are equivalent.

Various properties of multiple-free algebras which, in view of 8.13, can be derived directly from theorems on finite and multiple-free elements in §4 will not be stated explicitly here. By applying these properties, we obtain, e.g.,

THEOREM 8.14. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a multiple-free G.C.A., then* $\breve{\mathfrak{A}}$ *is an idem-multiple G.C.A.*

PROOF: By 4.37 and 8.11, $\mathfrak{A}$ satisfies the hypothesis, and hence also the conclusion, of 8.10.

THEOREM 8.15. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a G.C.A. and if* B *is the set of all multiple-free elements in* A, *then* $\mathfrak{B} = \langle B, +, \sum \rangle$ *is a multiple-free generalized cardinal subalgebra of* $\mathfrak{A}$.

PROOF: By 4.29, the set B satisfies the condition: if $a \in B$, $b \in A$, and $b \leqq a$, then $b \in B$. Hence Postulates 5.1.I, II, IV, V hold in $\mathfrak{B}$, since all elements whose existence is claimed in these postulates are at most equal to certain elements whose existence is assumed there. By 4.28, B contains the element 0, and hence 5.1.III is satisfied in $\mathfrak{B}$. Thus, by 6.13, $\mathfrak{B}$ is a generalized cardinal subalgebra of $\mathfrak{A}$. From 4.26 we see that every element which is multiple-free in $\mathfrak{A}$ is also multiple-free in $\mathfrak{B}$. Hence, by 8.13, $\mathfrak{B}$ is multiple-free; and the proof is complete.

One property of multiple-free algebras deserves attention from the axiomatic point of view. It is easily seen by 3.13, 3.24, and 8.13 (i),(iii) that the elements a_ι, b_i, and c which satisfy the conclusions of the existential postulates 5.1.IV, V are uniquely determined in every multiple-free algebra. In consequence, various theorems whose proof in the general case requires an application of the axiom of choice can be established for multiple-free G.C.A.'s without the help of this axiom. All arithmetical theorems of Part I, except 3.35 and 3.36, belong here, as well as the imbedding theorem 7.8. It might be interesting in this connection to analyze the proofs of such theorems as 2.1, 2.6, 2.21, and 2.31 (where the method of iteration is used). It should be mentioned that, in general, the arithmetic of multiple-free algebras is much simpler than that of arbitrary G.C.A.'s; this follows if only from the fact that all elements in multiple-free algebras are finite, and that consequently the cancellation law for sums holds (cf. 4.19, 4.27, and 8.13).

It is easily seen that isomorphic images, subalgebras, and cardinal products of multiple-free G.C.A.'s are again multiple-free G.C.A.'s. However, this by no means applies to coset algebras of multiple-free G.C.A.'s, which may have the most diverse structures. In fact, as we shall see, every C.A. is isomorphic with a coset algebra of a multiple-free G.C.A. (see the remarks which follow Theorem 16.4 below).

DEFINITION 8.16. *Given an algebra* $\mathfrak{A} = \langle A, +, \sum \rangle$, *we define the operations* $\dot{+}$ *and* $\dot{\sum}$ *of* DISJUNCTIVE ADDITION *by putting for any elements* $a, b, a_0, a_1, \cdots, a_i, \cdots \varepsilon A$:

(i) $\quad a \dot{+} b = a + b \quad$ *in case* $\quad a + b \,\varepsilon\, A \quad$ *and* $\quad a \cap b = 0$;

(ii) $\quad \dot{\sum}_{i<\infty} a_i = \sum_{i<\infty} a_i \quad$ *in case* $\quad \sum_{i<\infty} a_i \,\varepsilon\, A \quad$ *and* $\quad a_i \cap a_j = 0 \quad$ *for all* i *and* j *with* $i < j < \infty$; *otherwise, the operations* $\dot{+}$ *and* $\dot{\sum}$ *are not defined.*

Furthermore, we put:

(iii) $$\dot{\mathfrak{A}} = \langle A, \dot{+}, \dot{\sum} \rangle.$$

$\dot{\mathfrak{A}}$ *is called the* DISJUNCTIVE ALGEBRA CORRELATED WITH $\mathfrak{A}$.

THEOREM 8.17. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a G.C.A., then* $\dot{\mathfrak{A}}$ *is a multiple-free G.C.A., and the zero elements in* $\mathfrak{A}$ *and* $\dot{\mathfrak{A}}$ *coincide.*

PROOF: With the help of 3.1, 3.12, and 8.16 we easily show that Postulates 5.1.I–V remain valid if we replace everywhere in them '$+$' and '$\sum$' by '$\dot{+}$' and '$\dot{\sum}$', and that consequently $\dot{\mathfrak{A}}$ is a G.C.A. To show that $\dot{\mathfrak{A}}$ is multiple-free, we notice that

$$a \cap b = 0 \quad \text{implies} \quad a \,\dot{\cap}\, b = 0$$

for any $a, b \,\varepsilon\, A$, and we apply 8.13(i),(iii).

COROLLARY 8.18. *For a G.C.A.* $\mathfrak{A}$ *to be multiple-free it is necessary and sufficient that* $\mathfrak{A} = \dot{\mathfrak{A}}$.

PROOF: by 8.13(i),(iii), 8.16, and 8.17.

It should be emphasized that the disjunctive algebra $\dot{\mathfrak{A}}$ of a G.C.A. $\mathfrak{A}$ is by no means a subalgebra of $\mathfrak{A}$. If $\mathfrak{A}$ is a C.A., then $\dot{\mathfrak{A}}$ is a G.C.A., but never a C.A. (unless $\mathfrak{A}$ has only one element—cf. 8.12 and 8.17). By means of $\dot{\mathfrak{A}}$, however, we can construct new C.A.'s. For instance, the closure $\overline{\dot{\mathfrak{A}}}$ of $\dot{\mathfrak{A}}$ is of course a C.A., though its structure may differ considerably from that of the original C.A. $\mathfrak{A}$. In our further discussion we shall find important examples of C.A.'s among coset algebras $\dot{\mathfrak{A}}/R$ generated by infinitely additive equivalence relations R. This throws some light on the introductory remarks in §5 regarding the significance of G.C.A.'s for the study of C.A.'s.

THEOREM 8.19. *For every algebra* $\mathfrak{A} = \langle A, +, \sum \rangle$ *the following conditions are equivalent:*

(i) *$\mathfrak{A}$ is an idem-multiple G.C.A., and for any $a, b \varepsilon A$ with $a \leqq b$ there is an element $c \varepsilon A$ such that $a + c = b$ and $a \cap c = 0$;*

(ii) *$\dot{\mathfrak{A}}$ is a G.C.A. and $\mathfrak{A} = \breve{\dot{\mathfrak{A}}}$;*

(iii) *there is a multiple-free G.C.A. $\mathfrak{B}$ such that $\mathfrak{A} = \breve{\mathfrak{B}}$.*

Instead of requiring in (i) *that $\mathfrak{A}$ be idem-multiple, we can require only that $2 \cdot a$ be in A for every $a \varepsilon A$.*

Proof: If (i) holds, we have by 8.9

(1) $$\mathfrak{A} = \breve{\mathfrak{A}}.$$

Moreover, (i) implies by 8.16 that the relations $\leqq$ in $\mathfrak{A}$ and $\dot{\leqq}$ in $\dot{\mathfrak{A}}$ coincide; hence we obtain by 3.2 and 8.7

(2) $$\breve{\mathfrak{A}} = \breve{\dot{\mathfrak{A}}}.$$

Theorem 8.17 and formulas (1) and (2) at once give (ii). From (ii) we easily conclude that $\dot{\mathfrak{A}}$ is multiple-free (by applying 8.11 and 8.16, and without assuming that $\mathfrak{A}$ is itself a G.C.A.); and hence we obtain (iii). By 8.14, condition (iii) implies that $\mathfrak{A}$ is an idem-multiple G.C.A.; and from 8.8 and 8.13(i),(iii) we see that $\mathfrak{A}$ also satisfies the additional condition:

(3) if $a, b \varepsilon A$ and $a \leqq b$, then there is an element $c \varepsilon A$ such that $a + c = b$ and $a \cap c = 0$.

Thus, conditions (i)–(iii) are equivalent. Assume finally that $\mathfrak{A}$ is a G.C.A. which satisfies (3) and in which $2 \cdot a \varepsilon A$ for every $a \varepsilon A$. In this case, for every element $a \varepsilon A$, there is an element $c \varepsilon A$ such that

$$a + c = 2 \cdot a \quad \text{and} \quad a \cap c = 0.$$

Hence, by 3.12,

$$c = (2 \cdot a) \cap c = 0 \quad \text{and} \quad a = 2 \cdot a,$$

so that, by 8.1, $\mathfrak{A}$ is idem-multiple. If, conversely, $\mathfrak{A}$ is idem-multiple then obviously $2 \cdot a \varepsilon A$ for every $a \varepsilon A$. Thus, the last part of the theorem has been established.

Theorem 8.20. *For every algebra $\mathfrak{A} = \langle A, +, \sum \rangle$ the following conditions are equivalent:*

(i) *$\mathfrak{A}$ is a multiple-free G.C.A.;*

(ii) *$\breve{\mathfrak{A}}$ is a G.C.A. and $\mathfrak{A} = \dot{\breve{\mathfrak{A}}}$;*

(iii) *there is a G.C.A. $\mathfrak{B}$ such that $\mathfrak{A} = \dot{\mathfrak{B}}$.*

From the G.C.A. $\mathfrak{B}$ *in* (iii) *we can require that it satisfy condition* 8.19(i) *for* $\mathfrak{A} = \mathfrak{B}$.

PROOF. If (i) holds, then, by 8.14, $\breve{\mathfrak{A}}$ is a G.C.A.; and from 3.23, 8.8, 8.13, and 8.16 we conclude that the operations $+$ and $\dot{\cup}$ as well as $\sum$ and $\dot{\bigcup}$ coincide; thus, we obtain (ii). Conversely, (ii) implies (i) by 8.17. If (i) and (ii) hold, and if we put

$$\mathfrak{B} = \breve{\mathfrak{A}},$$

we at once obtain (iii); and we conclude from 8.19 that $\mathfrak{B}$ satisfies 8.19(i). Finally, (iii) implies (i), and hence also (ii), by 8.17. This completes the proof.

Theorems 8.19 and 8.20 exhibit close connections between multiple-free G.C.A.'s and those idem-multiple G.C.A.'s which satisfy the additional condition of 8.19(i). Given an algebra $\mathfrak{A}$ of the first class, we obtain an algebra $\mathfrak{B}$ of the second class by putting

$$\mathfrak{B} = \breve{\mathfrak{A}};$$

conversely, every algebra $\mathfrak{A}$ of the second class yields an algebra $\mathfrak{B}$ of the first class, namely,

$$\mathfrak{B} = \dot{\mathfrak{A}}.$$

We could say that we are presented in both cases with the same algebras, which are characterized, however, in terms of different fundamental operations. (The phrase 'the same algebras' should not be taken literally.)

Idem-multiple and multiple-free algebras exhibit, aside from many differences, also some striking similarities. For instance, in both kinds of algebras the sum of any sequence of elements, if it exists at all, coincides with the least upper bound of these elements; and the latter always exists if the sequence is bounded above. The first of these properties plays an essential role in the proof of the following theorem, which concludes this section:

THEOREM 8.21. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *and* $\mathfrak{A}' = \langle A', +', \sum' \rangle$ *be two idem-multiple or multiple-free G.C.A.'s. For a function f to map* $\mathfrak{A}$ *isomorphically onto* $\mathfrak{A}'$ *it is necessary and sufficient that* $D(f) = A$, $C(f) = A'$, *and that the formulas*

$$a \leqq b \quad \text{and} \quad f(a) \leqq' f(b)$$

be equivalent for any a and b in A.

Proof: The necessity of the conditions follows at once from 6.1. To show that they are sufficient it suffices to derive from them 6.1 (iii), and to apply 6.2. The derivation of 6.1(iii), however, presents no difficulty, since in both kinds of algebras the operation $+$ can be defined in terms of the relation $\leq$; cf. 8.2 (or 7.12 and 8.1) for idem-multiple algebras, and 7.12 and 8.13(i),(iii) for multiple-free algebras.

§ 9. SEMI-IDEALS AND IDEALS

In this section we shall discuss certain special subalgebras of G.C.A.'s—SEMI-IDEALS and IDEALS.[3]

DEFINITION 9.1. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$.

(i) *A non-empty subset B of A is called a* SEMI-IDEAL IN $\mathfrak{A}$ *if it satisfies the condition:*

$$\text{if} \quad a \in A, \quad b \in B, \quad \text{and} \quad a \leqq b, \quad \text{then} \quad a \in B.$$

(ii) *A semi-ideal C is called an* IDEAL IN $\mathfrak{A}$ *if it satisfies the condition:*

if $a \in A$, $c_0, c_1, \cdots, c_i, \cdots \in C$, *and* $a = \sum_{i<\infty} c_i$, *then* $a \in C$.

We are not going to consider here IDEALS IN A WIDER SENSE, i.e., semi-ideals which are assumed to be (relatively) closed under binary addition only. In opposition to ideals in a wider sense, those actually defined in 9.1(ii) could be called INFINITELY ADDITIVE (or INFINITELY CLOSED) IDEALS.

As examples of semi-ideals we may mention the sets of all finite and of all multiple-free elements; also the set of all indecomposable elements with 0 included. By applying to these semi-ideals the construction described in Theorem 9.7 below, we obtain, as examples of ideals, the set of all infinite sums of finite elements, the set of all infinite sums of multiple-free elements, and the set of all finite and infinite sums of indecomposable elements.

COROLLARY 9.2. *Every semi-ideal and every ideal in a G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$ *contains the element* 0.

PROOF: by 9.1.

COROLLARY 9.3. *Let C be an ideal in a G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$. *If* $n \leqq \infty$, $\sum_{i<n} c_i$ *is in* A, *and* $c_0, c_1, \cdots, c_i, \cdots$ *are in* C, *then* $\sum_{i<n} c_i$ *is in* C.

PROOF: by 9.1(ii) and 9.2.

COROLLARY 9.4. *The largest semi-ideal and the largest ideal in a G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is the set* A.

PROOF: by 9.1.

[3] The discussion in §§9 and 10 shows various analogies to the discussion of ideals in other algebraic systems, especially in lattices.

THEOREM 9.5. *If* K *is a non-empty family of semi-ideals in a G.C.A.* $\mathfrak{A}$, *then the union and the intersection of all sets in* K *are semi-ideals in* $\mathfrak{A}$.

PROOF: by 9.1(i) and 9.2.

THEOREM 9.6. *If* K *is a non-empty family of ideals in a G.C.A.* $\mathfrak{A}$, *then the intersection of all sets in* K *is an ideal in* $\mathfrak{A}$.

PROOF: by 9.1 and 9.2.

THEOREM 9.7. *If B is a semi-ideal in a G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$, *then the set C of all elements $c \varepsilon A$ such that*

$$c = \sum_{i<\infty} b_i \quad \textit{for some} \quad b_0, b_1, \cdots, b_i, \cdots \varepsilon B$$

is the smallest ideal in $\mathfrak{A}$ *which includes B.*

PROOF: by 9.1, with the help of 1.42 and 2.2.

THEOREM 9.8. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A., and let D be a subset of A.*

(i) *The set B, consisting of all elements $b \varepsilon A$ with $b \leqq d$ for some $d \varepsilon D$, and of the element 0 in case D is empty, is the smallest semi-ideal in* $\mathfrak{A}$ *which includes D.*

(ii) *The set C of all elements $c \varepsilon A$ such that*

$$c = \sum_{i<\infty} b_i \quad \textit{for some} \quad b_0, b_1, \cdots, b_i, \cdots \varepsilon B$$

is the smallest ideal in $\mathfrak{A}$ *which includes D.*

PROOF: by 9.1, 9.2, and 9.7.

THEOREM 9.9. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a C.A. and if D is a non-empty subset of A, then the set C of all elements $c \varepsilon A$ such that*

$$c \leqq \sum_{i<\infty} d_i \quad \textit{for some} \quad d_0, d_1, \cdots, d_i, \cdots \varepsilon D$$

is the smallest ideal in $\mathfrak{A}$ *which includes D.*

PROOF: The identity of the sets C in 9.8 and 9.9 follows from 1.22 and 2.2.

In case the set D in 9.8 (or 9.9) consists of one element only, we obtain what may be called PRINCIPAL SEMI-IDEALS and IDEALS. We introduce in this connection a special notation:

DEFINITION 9.10. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A. The smallest semi-ideal in* $\mathfrak{A}$ *which contains an element* $a \in A$ *is denoted by '*$S(a)$*'; and the smallest ideal in* $\mathfrak{A}$ *which contains* a *is denoted by '*$I(a)$*'.*

THEOREM 9.11. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a G.C.A. and* $a \in A$*, then the semi-ideal* $S(a)$ *is the set of all elements* $c \in A$ *with* $c \leqq a$*, and the ideal* $I(a)$ *is the set of all elements* $c \in A$ *with*

$$c = \sum_{i<\infty} c_i$$

where $c_i \leqq a$ *for every* $i < \infty$*;* $I(a)$ *includes* $S(a)$*.*

PROOF: by 9.8 and 9.10.

COROLLARY 9.12. *If* $\mathfrak{A}$ *is a G.C.A., then* $S(0) = I(0)$ *is the set consisting only of* 0*, and it is the smallest semi-ideal and the smallest ideal in* $\mathfrak{A}$*.*

PROOF: by 9.2 and 9.11.

COROLLARY 9.13. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a G.C.A. and* $a, \infty \cdot a \in A$*, then* $I(a)$ *is the set of all elements* $c \in A$ *with* $c \leqq \infty \cdot a$*, and* $I(a) = I(\infty \cdot a) = S(\infty \cdot a)$*.*

PROOF: by 9.11, with the help of 1.22, 1.43, and 2.2.

COROLLARY 9.14. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a G.C.A. and* $a, c, \infty \cdot a \in A$*, then the formulas* $I(a) = I(c)$ *and* $\infty \cdot a = \infty \cdot c$ *are equivalent.*

PROOF: by 9.10 and 9.13, with the help of 1.23 and 1.43.

THEOREM 9.15. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a G.C.A. and* $a \in A$*, then the set* $A(a)$ *of all elements* $c \in A$ *with* $c + a = a$ *is an ideal which is included in* $S(a)$ *and* $I(a)$*.*

PROOF: by 1.30, 1.47, 9.1, and 9.11.

THEOREM 9.16. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a G.C.A., then each of the formulas:* $A(a) = S(a)$ *and* $A(a) = I(a)$ *(where* $A(a)$ *is the ideal of* 9.15*) is equivalent to the condition that* a *is idem-multiple. If, in addition,* $\infty \cdot a \in A$*, the same applies to the formula:* $S(a) = I(a)$*.*

PROOF: by 4.3, 4.4, 9.11, 9.13, and 9.15.

THEOREM 9.17. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A. If* $a \in A$*, then the set* $D(a)$ *of all elements* $d \in A$ *with* $a \cap d = 0$ *is an ideal in* $\mathfrak{A}$*. More generally, if* C *is a subset of* A*, then the set* $D(C)$ *of all elements* $d \in A$ *with* $c \cap d = 0$ *for every* $c \in C$ *is an ideal in* $\mathfrak{A}$*.*

Proof: From 3.1 and 9.1(i) it is seen at once that the set $D(a)$, or $D(C)$, is a semi-ideal in $\mathfrak{A}$; hence, by 3.12 and 9.1(ii), it is an ideal.

Further theorems concern relations between semi-ideals, ideals, and arbitrary subalgebras of G.C.A.'s.

Theorem 9.18. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$.

(i) *If* $\mathfrak{A}$ *is a G.C.A. and* B *is a semi-ideal (or an ideal) in* $\mathfrak{A}$, *then* $\mathfrak{B} = \langle B, +, \sum \rangle$ *is a generalized cardinal subalgebra of* $\mathfrak{A}$.

(ii) *If* $\mathfrak{A}$ *is a C.A. and* B *is an ideal in* $\mathfrak{A}$, *then* $\mathfrak{B} = \langle B, +, \sum \rangle$ *is a cardinal subalgebra of* $\mathfrak{A}$.

In both cases $\mathfrak{A}$ *and* $\mathfrak{B}$ *have the common zero element* 0.

Proof: (i) by 5.1, 6.13, 9.1, and 9.2; (ii) by (i), 5.24, 6.13, and 9.1. In connection with (i) compare the proof of Theorem 8.15, which is a particular case of 9.18(i).

Regarded as subalgebras, semi-ideals and ideals deserve attention from the following point of view. Although the fundamental operations in an algebra $\mathfrak{A} = \langle A, +, \sum \rangle$ and in its subalgebra $\mathfrak{B} = \langle B, +, \sum \rangle$ coincide, other arithmetical notions discussed in Part I may have different meanings in $\mathfrak{A}$ and in $\mathfrak{B}$, even when applied to elements of $\mathfrak{B}$. For instance, $\mathfrak{A}$ and $\mathfrak{B}$ may have different zero elements; a and b being two elements of B, the formula $a \leqq b$ may hold in $\mathfrak{A}$, but not in $\mathfrak{B}$; and so on. This cannot occur, however, if B is a semi-ideal. For, as we know, $\mathfrak{A}$ and $\mathfrak{B}$ have in this case the same zero element; if, for any elements a, b, a_0, a_1, $\cdots$ in B, a formula like

$$a \leqq b, \quad a = \bigcap_{i<\infty} a_i \quad (\text{with } n \neq 0), \quad \text{or} \quad a = \bigcup_{i<\infty} a_i$$

holds in one of the algebras $\mathfrak{A}$ and $\mathfrak{B}$, it also holds in the other; and if an element a in B is, e.g., finite in $\mathfrak{A}$, it is also finite in $\mathfrak{B}$, and conversely. The proof presents no difficulty.

Theorem 9.19. *Let* $\mathfrak{B} = \langle B, +, \sum \rangle$ *be a subalgebra of a G.C.A* $\mathfrak{A} = \langle A, +, \sum \rangle$.

(i) *If* B *is a semi-ideal in* $\mathfrak{A}$ *and* D *is a semi-ideal in* $\mathfrak{B}$, *then* D *is also a semi-ideal in* $\mathfrak{A}$; *similarly for ideals.*

(ii) *If* B *is a semi-ideal in* $\mathfrak{A}$ *and* D *is an ideal in* $\mathfrak{B}$, *then there is an ideal* C *in* $\mathfrak{A}$ *whose intersection with* B *is* D; *e.g., the smallest ideal in* $\mathfrak{A}$ *which includes* D *can be taken for* C.

Proof: (i) by 9.1 (and 6.13); (ii) by 9.1, 9.2, and 9.8.

It may be noticed that both parts (i) and (ii) of 9.19 may fail in case $\mathfrak{B}$ is a generalized cardinal subalgebra of $\mathfrak{A}$, but B is not a semi-ideal in $\mathfrak{A}$. We have, however, in this connection

THEOREM 9.20. *The conclusion of* 9.19(ii) *holds also in case* $\mathfrak{B}$ *is an arbitrary cardinal subalgebra of a C.A.* $\mathfrak{A}$, *and* D *is an ideal in* $\mathfrak{B}$.

PROOF: by 6.13, 9.1, and 9.9.

THEOREM 9.21. *Let* B *be a semi-ideal in a G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$; *or, more generally, let* $\mathfrak{B} = \langle B, +, \sum \rangle$ *be a subalgebra of* $\mathfrak{A}$ *which has the same zero element as* $\mathfrak{A}$.

(i) *If* C *is a semi-ideal (or an ideal) in* $\mathfrak{A}$, *then the intersection of* C *and* B *is a semi-ideal (or an ideal) in* $\mathfrak{B}$.

(ii) *If, in addition,* C *is included in* B, *then* C *is itself a semi-ideal (or an ideal) in* $\mathfrak{B}$.

PROOF: by 6.13, 9.1, 9.2, and 9.18.

THEOREM 9.22. *If* $\bar{\mathfrak{A}} = \langle \bar{A}, +, \sum \rangle$ *is a closure of* $\mathfrak{A} = \langle A, +, \sum \rangle$, *then* A *is a semi-ideal in* $\bar{\mathfrak{A}}$, *and* $\bar{A}$ *is the smallest ideal in* $\bar{\mathfrak{A}}$ *which includes* A.

PROOF: by 7.1, 7.4, and 9.1.

THEOREM 9.23. *If* $\mathfrak{B} = \langle B, +, \sum \rangle$ *is a subalgebra of a C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$ *constituted by a semi-ideal* B *in* $\mathfrak{A}$, *then there is a uniqely determined subalgebra* $\bar{\mathfrak{B}}$ *of* $\mathfrak{A}$ *which is a closure of* $\mathfrak{B}$; *in fact,* $\bar{\mathfrak{B}} = \langle \bar{B}, +, \sum \rangle$ *where* $\bar{B}$ *is the smallest ideal in* $\mathfrak{A}$ *(and also the smallest cardinal subalgebra of* $\mathfrak{A}$*) which includes* B.

PROOF: by 6.13, 7.1, 9.7, and 9.18.

In view of 7.8, 9.22, and 9.23, we could simply define G.C.A.'s as subalgebras of C.A.'s constituted by semi-ideals.

THEOREM 9.24. *Let* $\bar{\mathfrak{A}}$ *be a closure of* $\mathfrak{A} = \langle A, +, \sum \rangle$. *If* C *is an ideal in* $\mathfrak{A}$, *then there is a uniquely determined ideal* $\bar{C}$ *in* $\bar{\mathfrak{A}}$ *whose intersection with* A *is* C; *in fact,* $\bar{C}$ *is the smallest ideal in* $\bar{\mathfrak{A}}$ *which includes* C.

PROOF: By 7.1(i), 9.19(ii), and 9.22, the smallest ideal $\bar{C}$ in $\bar{\mathfrak{A}}$ which includes C has the desired property; if B is any other ideal in $\bar{\mathfrak{A}}$ with the same property, it can easily be shown to be identical with $\bar{C}$ by means of 7.1(ii),(iii) and 9.1.

THEOREM 9.25. *If* C *is a semi-ideal, or an ideal, in a G.C.A.* $\mathfrak{A}$, *it is also a semi-ideal, or an ideal, in the disjunctive G.C.A.* $\dot{\mathfrak{A}}$.

Proof: by 8.16, 8.17, and 9.1.

Ideals provide us with a rather general method of contructing infinitely additive and infinitely refining equivalence relations. This method is based upon the following:

Definition 9.26. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A., and let* C *be an ideal in* $\mathfrak{A}$. *Two elements* $a, b \varepsilon A$ *are called* CONGRUENT MODULO C, *in symbols,* $a \underset{C}{\equiv} b$, *if there are elements* $a', b' \varepsilon C$ *and* $c \varepsilon A$ *such that* $a = a' + c$ *and* $b = b' + c$.

Theorem 9.27. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a C.A., and let* C *be an ideal in* $\mathfrak{A}$. *Then for any* $a, b \varepsilon A$ *the following conditions are equivalent:*

(i) $a \underset{C}{\equiv} b$;

(ii) *there are elements* $a', b' \varepsilon C$ *for which* $a + b' = b + a'$;

(iii) *there is an element* $c \varepsilon C$ *with* $a + c = b + c$.

Proof: (ii) follows directly from (i) by 9.26. To derive (iii) from (ii), we put

$$c = \infty \cdot a' + \infty \cdot b',$$

and apply 9.1(ii). (i) can be derived from (iii) by means of 2.6 (or 2.3) and 9.1(i).

Theorem 9.28. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a C.A., and let* $a, b, c \varepsilon A$.

(i) *If* $C = I(c)$, *we have* $a \underset{C}{\equiv} b$ *if, and only if,* $a + \infty \cdot c = b + \infty \cdot c$.

(ii) *If* $C = A(c)$, *where* $A(c)$ *is the ideal of* 9.15, *we have* $a \underset{C}{\equiv} b$ *if, and only if,* $a + c = b + c$.

Proof: first part by 9.9 and 9.27; second part by 2.6, 9.15, and 9.27.

Theorem 9.29. *If* C *is an ideal in a C.A., or G.C.A.,* $\mathfrak{A}$, *then* $\underset{C}{\equiv}$ *is an infinitely additive and infinitely refining equivalence relation in* $\mathfrak{A}$, *and hence* $\mathfrak{A}/\underset{C}{\equiv}$ *is a C.A., or G.C.A.; moreover,* C *coincides with the coset of the zero element,* $0/\underset{C}{\equiv}$.

Proof: In the proof we apply 9.1 and 9.26 several times. The relation $\underset{C}{\equiv}$ is clearly reflexive and symmetric; by means of 2.3 we show that it is transitive. From 6.4 we easily see that it is infinitely additive; to show that it is infinitely refining, we make use of 1.1.VI and 6.7. Hence, $\mathfrak{A}/\underset{C}{\equiv}$ is by 6.10 a C.A., or G.C.A. The last part of the conclusion follows easily from 6.3.

From 9.28(ii) and 9.29 we can derive, e.g., with the help of 7.13, various properties of the relation R which holds between two ele-

ments a and b of a C.A. $\mathfrak{A}$ if, and only if, $a + c = b + c$ (where c is any fixed element of $\mathfrak{A}$). Most of these properties, however, have been established in Part I directly; for instance, Theorems 2.16 and 2.18 show that the relation in question is additive and refining.

The following theorem is a converse of 9.29.

THEOREM 9.30. *Let R be an infinitely additive equivalence relation in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$. If R is finitely or infinitely refining in $\mathfrak{A}$—or, more generally, if $\mathfrak{A}/R$ is a G.C.A.—then the coset $0/R$ is an ideal.*

PROOF: By 6.3, the set $0/R$ is a subset of A. If a is in $0/R$ and $b \leqq a$, we obtain, by applying 6.3–6.5 and by passing to cosets,

$$b/R \leqq a/R = 0/R;$$

and since $\mathfrak{A}/R$ is a G.C.A. and $0/R$ is its zero element (cf. 6.6 and 6.10), we conclude by 5.19 that

$$b/R = 0/R;$$

thus, b is in $0/R$, and $0/R$ is a semi-ideal by 9.1(i). From 6.3 and 6.4 we easily see that $0/R$ satisfies also the condition stated in 9.1 (ii), and hence is an ideal.

It should be emphasized that the method of constructing infinitely additive and infinitely refining equivalence relations with the help of ideals, as provided by 9.29, is by no means universal. In fact, two different relations R and S with these properties can generate the same ideal (in the sense of 9.30); e.g., if R is any one of the relations $\widetilde{\sigma}$ which will be discussed in §11, then $0/R = I(0)$ (cf. 11.13 and 11.25).

In general, the variety of ideals in a C.A. or a G.C.A. can be very small, and in fact much smaller than the variety of elements. We see, for instance, from 9.13 that the ideals generated by two elements a and $\infty \cdot a$ are always equal, even if the elements themselves are different; and this will be further emphasized by Theorem 10.13 below which shows that the variety of ideals in an arbitrary C.A. $\mathfrak{A}$ is the same as that in a special subalgebra of $\mathfrak{A}$ constituted by all idem-multiple elements. We now want to consider the extreme case when an algebra has only two different ideals.

DEFINITION 9.31. *A G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$ is called* SIMPLE *if there are no ideals in $\mathfrak{A}$ different from A and $I(0)$.*

Examples of simple algebras will be found in §14.

THEOREM 9.32. *For every G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$ *the following conditions are equivalent:*

(i) $\mathfrak{A}$ *is simple;*

(ii) $I(a) = I(b)$ *for any* $a, b \,\varepsilon\, A$ *with* $a \neq 0$ *and* $b \neq 0$;

(iii) *if* $a, b \,\varepsilon\, A$ *and* $b \neq 0$, *then there are elements* $a_0, a_1, \cdots, a_i, \cdots \varepsilon\, A$ *such that*

$$a = \sum_{i<\infty} a_i$$

and $a_i \leqq b$ *for every* $i < \infty$.

PROOF: (i) clearly implies (ii) by 9.11, 9.12, and 9.31. If (i) does not hold, there is an ideal C different from A and $I(0)$. We consider an element a in C different from 0, and an element b in A which is not in C; and we easily see from 9.10 and 9.11 that $I(a) \neq I(b)$ (since $I(a)$ is a subset of C), so that (ii) fails. Furthermore, in view of 9.11, (iii) amounts to saying that $a \,\varepsilon\, I(b)$ and $b \,\varepsilon\, I(a)$ for any $a \neq 0$ and $b \neq 0$; and this is clearly equivalent to (ii) by 9.10.

THEOREM 9.33. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a simple G.C.A., and* a *is an infinite element of* A, *then* a *is idem-multiple and is the unique infinite element of* A; *and we have* $b \leqq a$ *for every* $b \,\varepsilon\, A$.

PROOF: The ideals $A(a)$ of 9.15 and $I(a)$ are different from $I(0)$, as is seen from 4.10, 9.11, and 9.15; hence, by 9.31, they are identical. Consequently, by 9.16, a is idem-multiple, and therefore, by 4.3 and 9.32,

$$b \leqq \infty \cdot a = a \text{ for every } b \,\varepsilon\, A.$$

If b is also infinite, we have for the same reasons

$$a \leqq \infty \cdot b = b,$$

so that a and b must be identical.

THEOREM 9.34. *For a C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$ *to be simple it is necessary and sufficient that* A *contain no two different infinite elements.*

PROOF: The condition is necessary by 9.33 for arbitrary G.C.A.'s. If the condition is satisfied, we have, by 4.12,

$$\infty \cdot a = \infty \cdot b \quad \text{for any} \quad a, b \,\varepsilon\, A \quad \text{with} \quad a \neq 0 \quad \text{and} \quad b \neq 0;$$

hence by 9.14 we obtain 9.32(ii), so that $\mathfrak{A}$ is simple.

From examples in §14 it will be seen that the condition of 9.34 is not sufficient for $\mathfrak{A}$ to be simple in case $\mathfrak{A}$ is a G.C.A., and not a C.A.

THEOREM 9.35. *Let $\mathfrak{A} = \langle A, +, \sum \rangle$ be a G.C.A., and let a be an indecomposable element of A. For $\mathfrak{A}$ to be simple, it is necessary and sufficient that, for every $b \in A$, there be an integer $n \leqq \infty$ such that $b = n \cdot a$.*

PROOF: An elementary argument based upon 4.38 and 4.40 shows that the condition of the theorem is equivalent to 9.32(iii)—under the assumption that a is indecomposable.

COROLLARY 9.36. *If $\mathfrak{A} = \langle A, +, \sum \rangle$ is a simple G.C.A., then A contains no two different indecomposable elements.*

PROOF: by 9.35 and 4.42.

THEOREM 9.37. *If $\mathfrak{A} = \langle A, +, \sum \rangle$ is a simple G.C.A., $0 < n < \infty$, $a_0, a_1, \cdots, a_i, \cdots \in A$, and $\bigcap_{i<n} a_i$ exists, then we have*

$$\bigcap_{i<n} a_i = a_m \quad \textit{for some} \quad m < n.$$

PROOF: We can put

$$\text{(1)} \qquad \bigcap_{i<n} a_i + b_j = a_j \quad \text{for} \quad j < n.$$

If $b_m = 0$ for some $m < n$, the conclusion obviously holds. If, on the contrary, $b_j \neq 0$ for every $j < n$, we show by induction with respect to n and with the help of 9.32(iii) that there is an element x such that

$$\text{(2)} \qquad x \neq 0, \quad \text{and} \quad x \leqq b_j \quad \text{for} \quad j < n.$$

From (1) and (2) we easily conclude that

$$\bigcap_{i<n} a_i + x \leqq a_j \quad \text{for} \quad j < n,$$

and hence

$$\bigcap_{i<n} a_i + x = \bigcap_{i<n} a_i .$$

Consequently, by 4.10, the element $\bigcap_{i<n} a_i$ is infinite. Hence, by 4.15 and 9.33, all the elements $a_0, a_1, \cdots, a_i, \cdots$ are infinite and equal to each other, so that the conclusion again holds.

THEOREM 9.38. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a simple G.C.A.,* $0 < n < \infty$, $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$, *and* $\bigcup_{i<n} a_i$ *exists, then we have*

$$\bigcup_{i<n} a_i = a_m \quad \text{for some} \quad m < n.$$

PROOF: By 9.33 the conclusion certainly holds if one of the elements a_i is infinite. Otherwise, $\bigcup_{i<n} a_i$ is finite by 4.17. We put

(1) $$a_j + b_j = \bigcup_{i<n} a_i \quad \text{for} \quad j < n.$$

By 4.21 and 9.37 we obtain

(2) $$\bigcup_{i<n} a_i = \bigcup_{i<n} a_i + \bigcap_{i<n} b_i = \bigcup_{i<n} a_i + b_m \quad \text{for some} \quad m < n.$$

(2) implies by 4.10 that $b_m = 0$. Hence, by (1), the conclusion.

COROLLARY 9.39. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a simple G.C.A., then for all* $a, b \varepsilon A$ *the following conditions are equivalent:*

(i) $a \leqq b$ *or* $b \leqq a$;
(ii) $a \cap b$ *exists;*
(iii) $a \cup b$ *exists.*

PROOF: by 9.37 and 9.38.

The problem remains open whether there are simple G.C.A.'s in which 9.39(i) fails for some a and b, i.e., which are not SIMPLY ORDERED.

§ 10. ALGEBRA OF IDEALS

We proceed to the construction of the ALGEBRA OF IDEALS.

DEFINITION 10.1. *Let $\mathfrak{A} = \langle A, +, \sum \rangle$. By the* SUM $B + C$ OF TWO IDEALS *B and C in $\mathfrak{A}$ we understand the smallest ideal in $\mathfrak{A}$ which includes both B and C; similarly we define the* SUM $\sum_{\iota<\infty} A_\iota$ OF AN INFINITE SEQUENCE OF IDEALS $A_0, A_1, \cdots, A_\iota, \cdots$ *in $\mathfrak{A}$. The algebra constituted by the family I of all ideals in $\mathfrak{A}$ and by the operations $+$ and $\sum$ just defined is called the* IDEAL ALGEBRA OF $\mathfrak{A}$, *in symbols,*

$$\mathfrak{J}(\mathfrak{A}) = \langle \mathsf{I}, +, \sum \rangle.$$

THEOREM 10.2. *Let $\mathfrak{A} = \langle A, +, \sum \rangle$ be a G.C.A. The sum $B + C$ of any two ideals B and C in $\mathfrak{A}$ always exists, and it consists of all elements $a \in A$ with $a = b + c$ for some $b \in B$ and $c \in C$. Similarly, the sum $\sum_{\iota<\infty} A_\iota$ of any infinite sequence of ideals $A_0, A_1, \cdots, A_\iota, \cdots$ in $\mathfrak{A}$ always exists, and it consists of all elements $a \in A$ with*

$$a = \sum_{\iota<\infty} a_\iota$$

where $a_\iota \in A_\iota$ for every $i < \infty$.

PROOF: B and C being any ideals in $\mathfrak{A}$, the set of all elements $b + c \in A$ with $b \in B$ and $c \in C$ is an ideal in $\mathfrak{A}$ by 2.4, 5.1.II, and 9.1; it is the smallest ideal which includes B and C by 9.2 and 9.3; and hence it coincides with $B + C$ by 10.1. Similarly for any infinite sequence of ideals, with 2.4 and 5.1.II replaced by 2.2 and 5.22.

THEOREM 10.3 (FUNDAMENTAL THEOREM OF THE IDEAL ALGEBRA). *If $\mathfrak{A} = \langle A, +, \sum \rangle$ is a G.C.A., then the ideal algebra $\mathfrak{J}(\mathfrak{A})$ is an idem-multiple C.A., and the ideal $I(0)$ is the zero element of this algebra.*

PROOF: From 10.1 and 10.2 we easily see that Postulates 1.1.I–IV hold in $\mathfrak{J}(\mathfrak{A})$. By 9.12 and 10.1 we have

$$B + I(0) = I(0) + B = B$$

for every ideal B in $\mathfrak{A}$; thus, Postulate 1.1.V is satisfied, and $I(0)$ is the zero element of $\mathfrak{A}$. To obtain 1.1.VI, we assume that

$$B + C = \sum_{i<\infty} A_i$$

for given ideals B, C, A_0, A_1, $\cdots$, A_i, $\cdots$. We define B_i to be the intersection of B and A_i; and similarly we define C_i. By 9.6, B_i and C_i are ideals. By 10.1, B is included in $\sum_{i<\infty} A_i$; hence, by 10.2, B consists of elements b with

$$b = \sum_{i<\infty} a_i \quad \text{where} \quad a_i \varepsilon A_i \quad \text{for} \quad i = 0, 1, 2, \cdots .$$

From 9.1 we see that these elements a_i are also in B. Hence

$$a_i \; \varepsilon \; B_i \quad \text{for} \quad i = 0, 1, 2, \cdots.$$

Therefore, by 10.2, B is included in the sum $\sum_{i<\infty} B_i$, whereas, by 10.1, it includes this sum; so that finally

$$B = \sum_{i<\infty} B_i.$$

Similarly we obtain

$$C = \sum_{i<\infty} C_i, \quad \text{and} \quad A_i = B_i + C_i \quad \text{for} \quad i = 0, 1, 2, \cdots .$$

Thus, the ideals B_i and C_i satisfy the conclusion of 1.1.VI.

Now to derive 1.1.VII, we consider ideals A_i and B_i such that

$$(1) \qquad A_n = B_n + A_{n+1} \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

From 10.1 we conclude by induction that A_{n+i} and B_{n+i} are subsets of A_n for every $i < \infty$. The intersection C of A_0, A_1, $\cdots$, A_i, $\cdots$ is an ideal by 9.6. Let a be any element in A_n. We put $a_0 = a$. By (1) and 10.2 we have

$$a_0 = b_0 + a_1 \quad \text{with} \quad b_0 \; \varepsilon \; B_n \quad \text{and} \quad a_1 \; \varepsilon \; A_{n+1} .$$

By continuing this procedure indefinitely, and then by applying 5.1.V (to elements a_0, a_1, $\cdots$, b_0, b_1, $\cdots$ ε C), we arrive at

$$(2) \quad a_p = c + \sum_{i<\infty} b_{p+i} \quad \text{where} \quad a_p \; \varepsilon \; A_{n+p} \quad \text{and} \quad b_p \; \varepsilon \; B_{n+p} \quad \text{for} \quad p = 0, 1, 2, \cdots .$$

Hence, by 9.1, c is in A_n, A_{n+1}, $\cdots$, A_{n+i}, $\cdots$ and therefore also in C (the sequence A_0, A_1, $\cdots$, A_i, $\cdots$ being decreasing). For $p = 0$, (2) gives

$$a = a_0 = c + \sum_{i<\infty} b_i ;$$

and this shows by 10.2 that a is in $C + \sum_{i<\infty} B_{n+i}$. Thus, A_n is included in $C + \sum_{i<\infty} B_{n+i}$. The sets C, B_n, B_{n+1}, $\cdots$ being subsets of A_n, the inclusion in the opposite direction results from 10.1, so that finally

$$A_n = C + \sum_{i<\infty} B_{n+i} \quad \text{for} \quad n = 0, 1, 2, \cdots ;$$

in other words, C satisfies the conclusion of 1.1.VII. Thus, $\mathfrak{J}(\mathfrak{A})$ is a C.A.; and from 8.1 and 10.1 it is seen that $\mathfrak{J}(\mathfrak{A})$ is an idem-multiple C.A.

THEOREM 10.4. *If $\mathfrak{A} = \langle A, +, \sum \rangle$ is a G.C.A., then the relation $\leqq$ in the ideal algebra $\mathfrak{J}(\mathfrak{A})$ coincides with set-theoretical inclusion; and the set A is the largest element in this algebra, i.e., we have $B \leqq A$ for every element B of the algebra.*

PROOF: by 9.4 and 10.1.

THEOREM 10.5. *Let $\mathfrak{A} = \langle A, +, \sum \rangle$ be a G.C.A., and let A_i be elements of the ideal algebra $\mathfrak{J}(\mathfrak{A})$ (i.e., ideals in $\mathfrak{A}$) correlated with elements i of an arbitrary set I. Then $\bigcap_{i\varepsilon I} A_i$ always exists; and, in case the set I is not empty, $\bigcap_{i\varepsilon I} A_i$ coincides with the set-theoretical intersection of all ideals A_i.*

PROOF: by 3.1, 9.6, 10.1, and 10.4.

COROLLARY 10.6. *For any ideals B and C in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$ the following conditions are equivalent:*

(i) *0 is the only common element of B and C;*

(ii) *$b \cap c = 0$ for every $b \varepsilon B$ and $c \varepsilon C$;*

(iii) *$B \cap C = I(0)$ in the ideal algebra $\mathfrak{J}(\mathfrak{A})$.*

PROOF: The equivalence of (i) and (ii) follows from 9.1 and 9.2; the equivalence of (i) and (iii) is a consequence of 9.12 and 10.5.

THEOREM 10.7. *Under the hypothesis of 10.5, $\bigcup_{i\varepsilon I} A_i$ always exists; it is the smallest ideal which includes all the ideals A_i, and it consists of all elements $a \varepsilon A$ which can be represented in the form*

$$a = \sum_{k<n} a_k$$

where $n \leqq \infty$, and $a_k \varepsilon A_{i_k}$ for some elements $i_k \varepsilon I$ such that $i_k \neq i_l$ for $k < l < n$.

PROOF: Let B be the set of all elements a which have the representation given in the theorem. By means of 1.44, 2.2, 9.1, and 9.3,

we easily show that B is an ideal, and in fact the smallest ideal which includes all ideals A_i. Hence, by 3.2 and 10.4,

$$B = \bigcup_{i \varepsilon I} A_i .$$

It is seen from 10.7 that the ideal $\bigcup_{i \varepsilon I} A_i$ is the set-theoretical union of all ideals $\bigcup_{i \varepsilon J} A_i$ where J is any at most denumerable subset of I.

Corollary 10.8. *If $\mathfrak{A} = \langle A, +, \sum \rangle$ is a G.C.A., $n \leqq \infty$, and $A_0, A_1, \cdots, A_i, \cdots$ with $i < n$ are ideals in $\mathfrak{A}$, then*

$$\bigcup_{i<n} A_i = \sum_{i<n} A_i ,$$

and $\bigcup_{i<n} A_i$ consists of all elements $a \varepsilon A$ of the form

$$a = \sum_{i<n} a_i$$

where $a_i \varepsilon A_i$ for every $i < n$.

Proof: by 8.2, 9.2, 10.3, and 10.7.

Theorem 10.9. *If B is an ideal in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$, and if C_i are ideals in $\mathfrak{A}$ correlated with elements $i \varepsilon I$, then*

$$B \cap \bigcup_{i \varepsilon I} C_i = \bigcup_{i \varepsilon I} (B \cap C_i).$$

Proof: By 10.1, 10.5, and 10.7, all the bounds involved exist in the algebra $\mathfrak{J}(\mathfrak{A})$ and are ideals in $\mathfrak{A}$. If, moreover,

$$x \varepsilon B \cap \bigcup_{i \varepsilon I} C_i , \tag{1}$$

we have by 10.5 and 10.7

$$x \varepsilon B, \quad \text{and} \quad x = \sum_{k<n} c_k \quad \text{where} \quad c_k \varepsilon C_{i_k} \quad \text{for} \quad k < n;$$

hence, by 9.1 and 10.5,

$$x = \sum_{k<n} c_k \quad \text{where} \quad c_k \varepsilon B \cap C_{i_k} \quad \text{for} \quad k < n;$$

and therefore, again by 10.7,

$$x \varepsilon \bigcup_{i \varepsilon I} (B \cap C_i). \tag{2}$$

Thus, (1) implies (2); by 10.4, this gives

$$B \cap \bigcup_{i \varepsilon I} C_i \leqq \bigcup_{i \varepsilon I} (B \cap C_i).$$

From 3.1 and 3.2 we easily obtain the inequality in the opposite direction, and hence finally the desired equation.

It should be noticed that the dual distributive law fails in general; it holds, of course, by 3.30 and 10.3 in case the set I is at most denumerable.

Theorems 10.3, 10.5, 10.7, and 10.9 provide us with an exhaustive characterization of the ideal algebra $\mathfrak{J}(\mathfrak{A})$. In §15 (Theorem 15.25) these results will be given a more convenient and familiar form.

THEOREM 10.10. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A. If* $n \leqq \infty$ *and* $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$, *then*

$$a = \sum_{i<n} a_i \quad \textit{implies} \quad I(a) = \sum_{i<n} I(a_i).$$

In case the elements $a, a_0, a_1, \cdots, a_i, \cdots$ *are idem-multiple, the implication in the opposite direction also holds.*

PROOF: If

$$(1) \qquad a = \sum_{i<n} a_i,$$

we see by 9.1, 9.3, and 9.10 that $I(a)$ is the smallest ideal which includes all the ideals $I(a_i)$; the existence of these ideals is secured by 9.11. Hence, by 10.1, we arrive at the conclusion.

If now $a, a_0, a_1, \cdots$ are idem-multiple and

$$I(a) = \sum_{i<n} I(a_i),$$

we conclude by 7.12, 9.10, and 10.8 that $\sum_{i<n} a_i$ is in A; hence by the first part of the theorem

$$I(a) = I(\sum_{i<n} a_i);$$

and by 4.3, 7.12, and 9.14 we obtain (1).

THEOREM 10.11. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be an idem-multiple G.C.A. Then the function* f *such that* $f(x) = I(x)$ *for every* $x \varepsilon A$ *maps* $\mathfrak{A}$ *isomorphically onto a generalized cardinal subalgebra* $\mathfrak{B}$ *of the ideal algebra* $\mathfrak{J}(\mathfrak{A})$.

PROOF: by 4.3, 6.1, 8.1, 9.10, 9.14, and 10.10.

Theorem 10.11 does not apply to arbitrary G.C.A.'s or even C.A.'s.

THEOREM 10.12. *If B is an ideal in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$, then the set of all ideals in $\mathfrak{B} = \langle B, +, \sum \rangle$ is an ideal in the ideal algebra $\mathfrak{J}(\mathfrak{A})$ (in fact, it is the smallest ideal which contains B); and $\mathfrak{J}(\mathfrak{B})$ is a cardinal subalgebra of $\mathfrak{J}(\mathfrak{A})$, with the same zero element $I(0)$.*

PROOF: by 9.18, 9.19, 9.21, and 10.1–10.4, without any difficulty.

THEOREM 10.13. *Let $\mathfrak{B} = \langle B, +, \sum \rangle$ be the subalgebra of a C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$ constituted by the set B of all idem-multiple elements in A; and let, for every ideal X in $\mathfrak{A}$, $F(X)$ be the intersection of X and B. Then F maps $\mathfrak{J}(\mathfrak{A})$ isomorphically onto $\mathfrak{J}(\mathfrak{B})$, so that*

$$\mathfrak{J}(\mathfrak{A}) \cong \mathfrak{J}(\mathfrak{B}).$$

PROOF: By 8.4, $\mathfrak{B}$ is an idem-multiple cardinal subalgebra of $\mathfrak{A}$, and by 4.2 the zero elements in $\mathfrak{A}$ and $\mathfrak{B}$ are identical. Hence, X being an ideal in $\mathfrak{A}$, $F(X)$ is by 9.21 an ideal in $\mathfrak{B}$. Conversely, Y being an ideal in $\mathfrak{B}$, 9.20 implies that there is an ideal X in $\mathfrak{A}$ such that

$$F(X) = Y.$$

Thus, $D(F)$ and $C(F)$ are the families of all ideals in $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Furthermore, for any ideals C and D in $\mathfrak{A}$, the formula

$$C \leqq D \tag{1}$$

implies

$$F(C) \leqq F(D); \tag{2}$$

we apply here 10.1, 10.4, and the definition of F. If now (2) holds, then, for the same reasons, D is an ideal which includes $F(C)$. Moreover, for every c in C, $\infty \cdot c$ is in C by 9.1(ii); it is also in B by 4.5, and therefore, by the definition of F, $\infty \cdot c$ is in $F(C)$ and in D; hence, by 9.1(i), c is in D. Consequently, (1) holds. Thus, formulas (1) and (2) are equivalent for all ideals in $\mathfrak{A}$. By now applying 8.21 and 10.3, we immediately arrive at the conclusion.

THEOREM 10.14. *Let $\bar{\mathfrak{A}}$ be a closure of its subalgebra $\mathfrak{A} = \langle A, +, \sum \rangle$; and let, for every ideal X in $\bar{\mathfrak{A}}$, $F(X)$ be the intersection of X and A. Then F maps $\mathfrak{J}(\bar{\mathfrak{A}})$ isomorphically onto $\mathfrak{J}(\mathfrak{A})$, so that*

$$\mathfrak{J}(\mathfrak{A}) \cong \mathfrak{J}(\bar{\mathfrak{A}}).$$

PROOF: The reasoning is entirely analogous to that in the preceding proof. Instead of 4.2 and 8.4 we apply 6.13, 7.1, and 7.2; and 9.24

shows at once that (2) implies (1), since C must coincide with the smallest ideal which includes $F(C)$.

Thus, instead of studying ideals in arbitrary G.C.A.'s, we can restrict ourselves to the discussion of ideals in C.A.'s (by 10.14), and even in special, idem-multiple C.A.'s (by 10.13).

COROLLARY 10.15. *If $\bar{\mathfrak{A}}$ is a closure of its subalgebra $\mathfrak{A}$, then $\mathfrak{A}$ is simple if, and only if, $\bar{\mathfrak{A}}$ is simple.*

PROOF: by 9.31, 10.1, and 10.14.

The last three theorems of this section will have a more special character; they will be applied in §12.

THEOREM 10.16. *Let $\mathfrak{A} = \langle A, +, \sum \rangle$ be a G.C.A. such that every element a in A can be represented in the form*

$$a = \sum_{i<n} a_i \quad \textit{with} \quad n \leqq \infty$$

where the elements a_i are indecomposable in the disjunctive algebra $\dot{\mathfrak{A}}$ —i.e., $a_i \neq 0$ and, for any $x, y \,\varepsilon\, A$, $a = x + y$ and $x \cap y = 0$ imply that $x = 0$ or $y = 0$. Then every ideal B in $\mathfrak{A}$ can be represented in the form

$$B = \bigcup_{i \varepsilon I} B_i \quad \textit{where} \quad B_i \cap B_j = I(0) \quad \textit{for} \quad i, j \,\varepsilon\, I \quad \textit{and} \quad i \neq j$$

and where the ideals B_i are indecomposable in the disjunctive ideal algebra $\mathfrak{J}(\mathfrak{A})$—i.e., $B_i \neq I(0)$ and, for any ideals X and Y in $\mathfrak{A}$, $B = X + Y$ and $X \cap Y = I(0)$ imply that $X = I(0)$ or $Y = I(0)$. This representation is unique apart from order.

PROOF: We shall consider first the particular case when $B = A$. Let K and L be families of ideals defined by the conditions:

(1) $X \,\varepsilon\, \mathsf{K}$ if, and only if, for some ideal Y in $\mathfrak{A}$, $A = X + Y$ and $X \cap Y = I(0)$;

(2) $X \,\varepsilon\, \mathsf{L}$ if, and only if, $X \,\varepsilon\, \mathsf{K}$ and X is indecomposable in $\mathfrak{J}(\mathfrak{A})$.

We notice the following:

(3) if z is indecomposable in $\dot{\mathfrak{A}}$, X and Y are ideals in $\mathfrak{A}$, $z \,\varepsilon\, X + Y$, and $X \cap Y = I(0)$, then $z \,\varepsilon\, X$ or $z \,\varepsilon\, Y$.

This is seen from 4.38, 8.16, 10.2, and 10.6. Hence and from (1) we conclude by 10.6:

(4) if z is indecomposable in $\dot{\mathfrak{A}}$ and $X \,\varepsilon\, \mathsf{K}$, then either $z \,\varepsilon\, X$ or $x \cap z = 0$ for every $x \,\varepsilon\, X$.

Given an element a indecomposable in $\mathfrak{A}$, there are ideals $X \varepsilon \mathsf{K}$ which contain a; e.g., as is seen from (1), 10.3, and 10.4, A is such an ideal. Let X_0 be the intersection of all ideals X with this property; and let Y_0 be the set of all elements y such that

$$x \cap y = 0 \quad \text{for every} \quad x \varepsilon X_0 .$$

By 9.17, Y_0 is an ideal; hence by 10.6

(5) $$X_0 \cap Y_0 = I(0).$$

Let y be any element which is indecomposable in $\mathfrak{A}$. If y is not in X_0, there is an ideal $X \varepsilon \mathsf{K}$ which includes X_0, but does not contain y. Hence, by (4), $x \cap y = 0$ for every $x \varepsilon X$ and *a fortiori* for every $x \varepsilon X_0$; and therefore y is in Y_0. Thus, every element which is indecomposable in $\mathfrak{A}$ is either in X_0 or in Y_0; by 9.3, 10.2, and the hypothesis of the theorem, we conclude hence that

(6) $$X_0 + Y_0 = A.$$

Furthermore, X_0 contains the indecomposable element a, and hence by 4.38, 8.17, and 9.12

(7) $$X_0 \neq I(0).$$

Assume now that, for some ideals X_1 and Y_1, we have

(8) $$X_0 = X_1 + Y_1 \quad \text{and} \quad X_1 \cap Y_1 = I(0).$$

By (3) and (8), a must belong to one of these ideals X_1 and Y_1, say, to X_1. From (5), (6), and (8) we easily obtain with the help of 3.12, 10.3, and 10.4

$$X_1 + (Y_1 + Y_0) = A \quad \text{and} \quad X_1 \cap (Y_1 + Y_0) = I(0);$$

thus, by (1), X_1 is in K. Therefore, by (8), 10.1, and the definition of X_0, the ideals X_0 and X_1 must be identical, and Y_1 must be equal to $I(0)$. Thus, (8) implies that

$$X_1 = I(0) \quad \text{or} \quad Y_1 = I(0);$$

and hence, by (7) and with the help of 4.38, 8.16, 8.17, 10.1, and 10.3, we conclude that

(9) $$X_0 \quad \text{is indecomposable in} \quad \mathfrak{J}(\mathfrak{A}).$$

From (1), (2), (5), (6), and (9), we see that $X_0 \varepsilon \mathsf{L}$.

We have thus shown that every element a indecomposable in $\dot{\mathfrak{A}}$ belongs to an ideal X_0 in L. Hence, by the hypothesis, every element a in A can be represented in the form

$$a = \sum_{i<n} a_i \quad \text{with} \quad n \leqq \infty$$

where each a_i belongs to a certain ideal $X_i \varepsilon \mathsf{L}$; by 9.3, we can replace in this representation all terms belonging to the same ideal X_i by their sum. By applying 10.7, we infer that

$$(10) \qquad A = \bigcup_{X \varepsilon \mathsf{L}} X.$$

Consider finally any two ideals $X_1, X_2 \varepsilon \mathsf{L}$. By (1), (2), 10.3, and 10.4, there is an ideal Y_1 with

$$X_2 \leqq X_1 + Y_1 \quad \text{and} \quad X_1 \cap Y_1 = I(0);$$

hence, by 3.13,

$$X_2 = (X_1 \cap X_2) + (Y_1 \cap X_2) \quad \text{and} \quad (X_1 \cap X_2) \cap (Y_1 \cap X_2) = I(0);$$

and since X_2 is indecomposable in $\dot{I}(\mathfrak{A})$, we conclude by 4.38 and 8.16 that

$$X_1 \cap X_2 = X_2 \quad \text{or} \quad X_1 \cap X_2 = I(0).$$

Similarly,

$$X_1 \cap X_2 = X_1 \quad \text{or} \quad X_1 \cap X_2 = I(0);$$

and consequently

$$(11) \qquad \text{if } X_1, X_2 \varepsilon \mathsf{L} \quad \text{and} \quad X_1 \neq X_2, \quad \text{then} \quad X_1 \cap X_2 = I(0).$$

From (2), (10), and (11) we see that the ideal $B = A$ has the representation required in the conclusion.

The result thus obtained can easily be extended to an arbitrary ideal B in $\mathfrak{A}$. For $\mathfrak{B} = \langle B, +, \sum \rangle$ is a G.C.A. by 6.13 and 9.18; and it obviously satisfies the hypothesis of the theorem (in view of 9.1). Hence, B has the desired representation in the ideal algebra $\mathfrak{J}(\mathfrak{B})$; and with the help of 9.1 and 10.12 we conclude that this will also be a representation in $\mathfrak{J}(\mathfrak{A})$.

Finally, assume B to have two such representations:

$$B = \bigcup_{i \varepsilon I} B_i = \bigcup_{j \varepsilon J} C_j .$$

We then obtain by 10.9

$$B_i = \bigcup_{j \varepsilon J} (B_i \cap C_j) \quad \text{for } i \varepsilon I, \quad \text{and} \quad C_j = \bigcup_{i \varepsilon I} (B_i \cap C_j) \quad \text{for} \quad j \varepsilon J.$$

Hence we easily conclude, with the help of 4.42 and 10.8, that the two representations of B differ at most in order (cf. the proof of 4.45 and the remarks preceding 1.38). This completes the proof.

Theorem 10.16 applies, in particular, to all those C.A.'s $\mathfrak{A}$ in which every element can be represented as a sum of elements which are indecomposable in $\mathfrak{A}$ itself (and not only in $\dot{\mathfrak{A}}$). The class of such C.A.'s, however, is much less comprehensive than that actually considered in 10.16. The latter contains, e.g., all C.A.'s $\mathfrak{A} = \langle A, +, \sum \rangle$ in which the set A is simply ordered by the relation $\leqq$ or, more generally, in which no two elements different from 0 are disjoint. It is, however, by no means true that in such an algebra $\mathfrak{A}$ every element is a sum of elements which are indecomposable in $\mathfrak{A}$.

Theorem 10.17. *Let $\mathfrak{A} = \langle A, +, \sum \rangle$ be a G.C.A. in which there is no infinite sequence of elements $b_0, b_1, \cdots, b_i, \cdots \varepsilon A$ with $b_i \neq 0$ and $b_i \cap b_j = 0$ for $i < j < \infty$. Then every ideal B in $\mathfrak{A}$ can be represented in the form*

$$B = \sum_{i<n} B_i \quad \textit{where} \quad n < \infty, \quad \textit{and} \quad B_i \cap B_j = I(0) \quad \textit{for} \quad i < j < n,$$

and where the ideals B_i are indecomposable in the disjunctive ideal algebra $\mathfrak{J}(\mathfrak{A})$; this representation is unique apart from order.

Proof: The algebra $\mathfrak{A}$ satisfies the hypothesis of 10.16, even in a stronger form, with '$n \leqq \infty$' changed to '$n < \infty$'. This is seen by applying 4.49 to the algebra $\dot{\mathfrak{A}}$ (in view of 8.16 and 8.17). Hence the conclusion of 10.16 holds. By 4.38, 8.17, and 10.3, all ideals B_i involved in this conclusion are different from $I(0)$. Hence the set of all these ideals must be finite, for otherwise, by 9.12 and 10.6, we should arrive at a contradiction of the hypothesis. By now applying 10.8, we obtain the conclusion of our theorem at once.

When speaking in our further discussion of finite, denumerable, etc., algebras $\mathfrak{A} = \langle A, +, \sum \rangle$, we shall have in mind that the sets A are finite, denumerable, etc.

THEOREM 10.18. *Every at most denumerable C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$ *satisfies the hypothesis, and hence also the conclusion, of* 10.17.

PROOF: If the hypothesis of 10.17 were not satisfied, we should have an infinite sequence of elements b_i with

$$b_i \neq 0 \quad \text{and} \quad b_i \cap b_j = 0 \quad \text{for} \quad i < j < \infty;$$

and we could form the set B of all elements b of the form

$$b = \sum_{i<\infty} b_{k_i}$$

where $k_0, k_1, \cdots, k_i, \cdots$ is an increasing sequence of non-negative integers without repetitions. We could then easily show by means of 3.12 that two elements b corresponding to two different sequences k_i are different; and that consequently the set B has the same power as the set of all sequences k_i, and cannot be denumerable. This completes the proof.

Theorem 10.18 cannot be extended to arbitrary G.C.A.'s; this can easily be shown by means of trivial examples of G.C.A.'s described in §5 (cf. the remarks which follow 5.2).

§ 11. PARTIAL AUTOMORPHISMS

There is a class of equivalence relations which, as we shall see in our further discussion, yields many interesting instances of C.A.'s and G.C.A.'s, especially when applied to multiple-free algebras. We arrive at these relations by considering PARTIAL AUTOMORPHISMS of G.C.A.'s. As is well known, an AUTOMORPHISM of a given algebra is a function which maps this algebra isomorphically onto itself. We define:

DEFINITION 11.1. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A. A function* f *is called a* PARTIAL AUTOMORPHISM IN $\mathfrak{A}$ *if* $D(f)$ *and* $C(f)$ *are semi-ideals in* $\mathfrak{A}$ *and if* f *maps* $\langle D(f), +, \sum \rangle$ *isomorphically onto* $\langle C(f), +, \sum \rangle$.

It would actually be sufficient for our purposes to consider only those partial automorphisms whose domain and counter-domain are principal semi-ideals, $S(a)$.

THEOREM 11.2. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A. The following conditions are necessary and sufficient for a function* f *to be a partial automorphism in* $\mathfrak{A}$:

(i) $D(f)$ *and* $C(f)$ *are semi-ideals in* $\mathfrak{A}$;

(ii) *for any elements* $a, b, c \in D(f)$ *the formulas*

$$a = b + c \quad \textit{and} \quad f(a) = f(b) + f(c)$$

are equivalent.

Condition (ii) *can be replaced by the following:*

(ii′) *for any elements* $a, a_0, a_1, \cdots, a_i, \cdots \in D(f)$ *the formulas*

$$a = \sum_{i<\infty} a_i \quad \textit{and} \quad f(a) = \sum_{i<\infty} f(a_i)$$

are equivalent.

PROOF: by 6.2, 6.13, 9.18, and 11.1.

COROLLARY 11.3. *If* f *is a partial automorphism in a G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$, *then*

(i) $f(0) = 0$;

(ii) *for any* $a \in A$ *and* $b \in D(f)$, *the formulas*

$$a \leqq b \quad \textit{and} \quad f(a) \leqq f(b)$$

are equivalent;

(iii) *for any* $a \,\varepsilon\, D(f)$ *and* $a_0, a_1, \cdots, a_i, \cdots \varepsilon\, A$ *with* $i < n \leqq \infty$, *the formulas*

$$a = \sum_{i<n} a_i \quad \textit{and} \quad f(a) = \sum_{i<n} f(a_i)$$

are equivalent.

PROOF: by 9.1 and 11.2.

COROLLARY 11.4. *Let* f *be a partial automorphism in a G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$.

(i) *If* $a \,\varepsilon\, D(f)$, $b \,\varepsilon\, A$, *and* $b \leqq f(a)$, *then there exists an element* $a' \,\varepsilon\, A$ *with* $a' \leqq a$ *and* $f(a') = b$.

(ii) *If* $n \leqq \infty$, $a \,\varepsilon\, D(f)$, $b_0, b_1, \cdots, b_i, \cdots \varepsilon\, A$, *and*

$$f(a) = \sum_{i<n} b_i,$$

then there exist elements $a_0, a_1, \cdots, a_i, \cdots \varepsilon\, A$ *such that*

$$a = \sum_{i<n} a_i, \quad \textit{and} \quad f(a_i) = b_i \quad \textit{for every} \quad i < n.$$

PROOF: by 9.1, 11.1, and 11.3.

COROLLARY 11.5. *If* $n \leqq \infty$ *and if* f *is a partial automorphism in a G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$, *then, for any* $a, a_0, a_1, \cdots, a_i, \cdots \varepsilon\, D(f)$,

(i) *the formulas*

$$a = \bigcup_{i<n} a_i \quad \textit{and} \quad f(a) = \bigcup_{i<n} f(a_i)$$

are equivalent;

(ii) *in case* $n \neq 0$, *the formulas*

$$a = \bigcap_{i<n} a_i \quad \textit{and} \quad f(a) = \bigcap_{i<n} f(a_i)$$

are also equivalent.

PROOF: by 3.1, (or 3.16), 3.17, 11.3, and 11.4.

We shall use the following notations applying to functions. The IDENTITY FUNCTION f, with $f(x) = x$ for every x, will be denoted by 'i'; we shall assume that the domain of i is restricted to elements of a given algebra $\mathfrak{A}$. The INVERSE OF A BIUNIQUE FUNCTION f will be denoted by 'f^{-1}'. Given a function f and a set B, we shall denote by '$f^{[B]}$' the function g obtained by restricting the domain of f to elements of B; i.e. $g(x) = f(x)$ if x is both in $D(f)$ and in B, and

$D(g)$ is the intersection of $D(f)$ and B. The COMPOSITION OF TWO FUNCTIONS f AND g, i.e., the function h such that $h(x) = f(g(x))$, will be denoted by 'fg'; the domain of fg is the set of all elements x such that $g(x)$ is in $D(f)$, while the counter-domain of fg is the set of all elements $f(x)$ with x in $C(g)$.

THEOREM 11.6. *Let $\mathfrak{A}$ be a G.C.A.*

(i) *The identity function i is a partial automorphism in $\mathfrak{A}$.*

(ii) *If f is a partial automorphism in $\mathfrak{A}$, the same applies to f^{-1}.*

(iii) *If f is a partial automorphism in $\mathfrak{A}$, and B is a semi-ideal in $\mathfrak{A}$, then $f^{[B]}$ is a partial automorphism in $\mathfrak{A}$.*

(iv) *If f and g are partial automorphisms in $\mathfrak{A}$, the same applies to fg.*

PROOF: by 6.1, 9.1, 9.4, 9.5, and 11.1–11.4.

The next few theorems concern extensions of partial automorphisms.

THEOREM 11.7. *If f is a partial automorphism in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$, then there is a uniquely determined partial automorphism g in $\mathfrak{A}$ with the following properties:*

(i) *$g(x) = f(x)$ for every $x \varepsilon D(f)$;*

(ii) *$D(g)$ consists of all elements $\sum_{i<n} a_i$ where $n \leqq \infty$, $a_0, a_1, \cdots, a_i, \cdots \varepsilon D(f)$, and both sums $\sum_{i<n} a_i$ and $\sum_{i<n} f(a_i)$ are in A;*

(iii) *$C(g)$ consists of all elements $\sum_{i<n} f(a_i)$ subjected to the same conditions.*

PROOF: Let B be the set of all elements $\sum_{i<n} a_i$ subjected to the conditions in (ii). Thus, for every $b \varepsilon B$, we have

$$b = \sum_{i<n} a_i \quad \text{with} \quad n \leqq \infty \quad \text{and} \quad a_0, a_1, \cdots, a_i, \cdots \varepsilon D(f);$$

and the element

$$c = \sum_{i<n} f(a_i)$$

is in A. By means of 1.44, 2.1, and 11.3(iii) we easily show that this element c depends solely on b (and not on the way in which a_0, $a_1, \cdots, a_i, \cdots$ have been chosen), so that we can put

$$c = g(b).$$

The remaining part of the proof is obvious; we apply 2.2, 9.1, and 11.2–11.4 several times.

COROLLARY 11.8. *If f is a partial automorphism in a C.A. $\mathfrak{A}$, then there is a uniquely determined partial automorphism g in $\mathfrak{A}$ with the following properties:*

(i) $g(x) = f(x)$ *for every* $x \varepsilon D(f)$;

(ii) $D(g)$ *is the smallest ideal in* $\mathfrak{A}$ *which includes* $D(f)$;

(iii) $C(g)$ *is the smallest ideal in* $\mathfrak{A}$ *which includes* $C(f)$.

PROOF: by 9.7, 11.1, and 11.7.

In our further discussion we shall often use operation and relation symbols provided with dots, like '$\dot{+}$,' '$\dot{\leqq}$,' and '$\dot{\cap}$.' According to the agreement at the beginning of §6, these symbols will denote operations and relations in the disjunctive algebra $\dot{\mathfrak{A}}$ correlated with a given G.C.A. $\mathfrak{A}$ (cf. 8.16).

THEOREM 11.9. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A., and let* n *be* $\leqq \infty$. *If* $f_0, f_1, \cdots, f_i, \cdots$ *with* $i < n$ *are partial automorphisms in* $\mathfrak{A}$, *and* $a, b, a_0, a_1, \cdots, a_i, \cdots$ *are elements in* A *such that*

(i) $a = \dot{\sum\limits_{i<n}} a_i, \quad b = \dot{\sum\limits_{i<n}} f_i(a_i), \quad$ *and* $\quad a_i \varepsilon D(f_i) \quad$ *for* $\quad i < n$,

then there is a uniquely determined partial automorphism f *in* $\mathfrak{A}$ *with*

(ii) $D(f) = S(a), \quad C(f) = S(b), \quad$ *and* $\quad f(x) = f_i(x) \quad$ *for* $\quad x \leqq a_i$ *and* $\quad i < n$.

PROOF: We shall apply Theorem 3.13 many times. By this theorem, 8.16, and the hypothesis, every element $x \leqq a$ can be uniquely represented in the form

$$x = \sum_{i<n} x_i \quad \text{with} \quad x_i \leqq a_i \quad \text{for} \quad i < n;$$

in fact,

$$x_i = x \cap a_i \quad \text{and} \quad x = \sum_{i<n} (x \cap a_i).$$

Similarly, by 3.13, 8.16, 11.3, and 11.4, every element $y \leqq b$ has a representation

$$y = \sum_{i<n} f_i(x \cap a_i) \quad \text{where} \quad x \leqq a.$$

In view of this, we define the function f by putting

(1) $$f(x) = \sum_{i<n} f_i(x \cap a_i) \quad \text{for every} \quad x \varepsilon S(a);$$

we conclude, with the help of 9.11 and the hypothesis, that

(2) $$D(f) = S(a) \quad \text{and} \quad C(f) = S(b),$$

and consequently

(3) $$D(f) \quad \text{and} \quad C(f) \quad \text{are semi-ideals.}$$

Furthermore, we show that for any $x, y, z \varepsilon D(f)$

(4) the formulas $x = y + z$ and $f(x) = f(y) + f(z)$ are equivalent.

In fact, from the first formula we get by 3.14, 8.16, 9.11, and (2)

$$x \cap a_i = (y \cap a_i) + (z \cap a_i) \quad \text{for every} \quad i < n,$$

and hence, by 9.1, 11.3(iii), and (1), we obtain the second formula; to derive the implication in the opposite direction, we apply 3.13 (several times), 11.3, and 11.4. By now applying 11.2 again, we conclude from (3) and (4) that

(5) f is a partial automorphism in $\mathfrak{A}$.

Moreover, we easily see by (1), 8.16, 9.11, and 11.3(i) that

(6) $$f(x) = f_i(x) \quad \text{for} \quad x \leqq a_i \quad \text{and} \quad i < n.$$

By (2), (5), and (6), f satisfies the conditions of the conclusion. Finally we show, by means of 3.13, 8.16, 9.11, and 11.2, that conditions (2), (5), and (6) imply (1). Thus, the function f which satisfies these conditions is uniquely determined; and the proof is complete.

THEOREM 11.10. *Let $\mathfrak{A} = \langle A, +, \sum \rangle$ be a finitely closed G.C.A. and let n be $< \infty$.*

(i) *If $f_0, f_1, \cdots, f_i, \cdots$ with $i < n$ are partial automorphisms in $\mathfrak{A}$ such that $D(f_i)$ and $C(f_i)$ are ideals with*

$$D(f_i) \cap D(f_j) = I(0) = C(f_i) \cap C(f_j) \quad \text{for} \quad i < j < n,$$

then there is a uniquely determined partial automorphism f in $\mathfrak{A}$ with the following properties:

$$D(f) = \sum_{i<n} D(f_i), \quad C(f) = \sum_{i<n} C(f_i), \quad \textit{and}$$

$$f(x) = f_i(x) \quad \textit{for} \quad x \,\varepsilon\, D(f_i) \quad \textit{and} \quad i < n.$$

(The symbols '$\cap$' and '$\sum$' refer to the ideal algebra $\mathfrak{I}(\mathfrak{A})$.)

(ii) *In case* $\mathfrak{A}$ *is a C.A.*, (i) *also holds for* $n = \infty$.

PROOF: The method is analogous to that used in the proof of 11.9. By means of 3.11, 10.6, and 10.8, we show that every element x in $\sum_{i<n} D(f_i)$ can be uniquely represented in the form

$$x = \sum_{i<n} x_i \quad \text{where} \quad x_i \,\varepsilon\, D(f_i) \quad \text{for every} \quad i < n;$$

and we define the function f by putting

$$f(x) = \sum_{i<n} f_i(x_i).$$

The notion of a finitely closed algebra has been defined in 5.26.

THEOREM 11.11. *If* f *is a partial automorphism in a G.C.A.* $\mathfrak{A}$, *it is also a partial automorphism in the correlated disjunctive G.C.A.* $\dot{\mathfrak{A}}$.

PROOF: by 8.16, 9.25, 11.2, 11.3(i), and 11.5(ii).

Our next theorem is of a more special and less elementary character than the theorems stated so far in this section; it will find some important applications in our further discussion.

THEOREM 11.12. *Let* f *and* g *be partial automorphisms in a G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$; *let* n *be any integer with* $0 < n \leqq \infty$, *and let* a_0, $a_1, \cdots, a_i, \cdots, b$, *and* c *be elements in* A *such that*

$$\dot{\sum_{i<n}} a_i, \ \dot{\sum_{i<n}} a_i \dot{+} c \,\varepsilon\, A \quad \textit{and} \quad f(\dot{\sum_{i<n}} a_i \dot{+} c) = b \dot{+} g(c).$$

Then there are elements $b_0, b_1, \cdots, b_i, \cdots, c_0, c_1, \cdots, c_i, \cdots$ *in* A *with*

$$b = \dot{\sum_{i<n}} b_i, \quad c = \dot{\sum_{i<n}} c_i, \quad \textit{and} \quad f(a_i \dot{+} c_i) = b_i \dot{+} g(c_i)$$

for every $i < n$.

PROOF: Our reasoning will be carried through within the algebra $\dot{\mathfrak{A}} = \langle A, \dot{+}, \dot{\sum} \rangle$ defined in 8.16; we must remember that $\dot{\mathfrak{A}}$ is a multiple-free G.C.A., and that f and g, as well as f^{-1} and g^{-1}, are partial automorphisms in this algebra; cf. 8.17, 11.6(ii), and 11.11. To simplify the argument slightly, we restrict ourselves to the case when $n = \infty$; it is easily seen by 11.3 that this does not involve any loss of generality.

For every i with $0 < i < \infty$ we define recursively a sequence of elements $c_{i,0}, c_{i,1}, \cdots, c_{i,k}, \cdots$ by putting

(1) $c_{i,0} = 0$, and $c_{i,k+1} = g^{-1}[g(c) \dot{\cap} f(a_i \dotplus c_{i,k})]$ for $k = 0, 1, 2, \cdots$.

Assume that, for a given $k < \infty$, all the elements $c_{i,k}$ exist and

(2) $$c_{i,k} \dot{\leqq} c.$$

Hence, by the hypothesis and 11.3(ii),

$$a_i \dotplus c_{i,k} \dot{\leqq} \dot{\sum_{i<n}} a_i \dotplus c \quad \text{and} \quad f(a_i \dotplus c_{i,k}) \dot{\leqq} b \dotplus g(c).$$

$\mathfrak{A}$ being multiple-free, we conclude by 4.37 and 8.13(i), (ii) (or else by 3.13 and 8.16) that the element

$$d_k = g(c) \dot{\cap} f(a_i \dotplus c_{i,k})$$

exists. By 11.4(i), d_k is in $C(g) = D(g^{-1})$; and therefore, by (1) and 11.3, $c_{i,k+1}$ exists and

(3) $$c_{i,k+1} \dot{\leqq} g^{-1}(g(c)) = c.$$

Thus, (2) implies (3); and since, by (1), (2) obviously holds for $k = 0$, it holds for every $k < \infty$. We also show by an easy induction with respect to k that

(4) $$c_{i,k} \dot{\leqq} c_{i,k+1} \quad \text{for} \quad 0 < i < \infty \quad \text{and} \quad k < \infty$$

and

(5) $$c_{i,k} \dot{\cap} c_{j,k} = 0 \quad \text{for} \quad 0 < i < j < \infty \quad \text{and} \quad k < \infty.$$

The proof rests upon (1), (2), 8.13 (or 8.16), 11.3, and 11.5. By applying 7.11, we conclude from (2) and (4) that the elements

(6) $$c_{i+1} = \dot{\bigcup_{k<\infty}} c_{i+1,k} \quad \text{for} \quad i < \infty$$

exist, and in fact that

(7) $$c_{i+1} \dot{\leqq} c.$$

(The same conclusion can be reached by means of (2), 4.36, and 8.13.) Formulas (4) and (5) imply

$$c_{i,k} \dot{\cap} c_{j,l} = 0 \quad \text{for} \quad 0 < i < j < \infty \quad \text{and} \quad k < l < \infty.$$

Hence and from (6) we easily obtain by 3.32

$$c_i \dot{\cap} c_j = 0 \quad \text{for} \quad 0 < i < j < \infty;$$

and, by applying 7.12, we get in view of (7)

$$\dot{\sum_{i<\infty}} c_{i+1} = \dot{\bigcup_{i<\infty}} c_{i+1} \dot{\leqq} c.$$

Consequently, there is an element c_0 such that

$$\text{(8)} \qquad c_0 \dot{+} \dot{\sum_{i<\infty}} c_{i+1} = \dot{\sum_{i<\infty}} c_i = c.$$

Furthermore, we show, by arguing as in the derivation of (3), that the element

$$g^{-1}[g(c) \dot{\cap} f(a_i \dot{+} c_i)]$$

exists for any given i with $0 < i < \infty$; and we derive from (1) and (6), by applying successively 3.26, 11.5(i), 3.32, and again 11.5(i),

$$g^{-1}[g(c) \dot{\cap} f(a_i \dot{+} c_i)] = \dot{\bigcup_{k<\infty}} g^{-1}[g(c) \dot{\cap} f(a_i \dot{+} c_{i,k})];$$

hence

$$g^{-1}[g(c) \dot{\cap} f(a_i \dot{+} c_i)] = \dot{\bigcup_{k<\infty}} c_{i,k+1} = c_i,$$

and finally

$$\text{(9)} \qquad g(c_i) = g(c) \dot{\cap} f(a_i \dot{+} c_i) \quad \text{for} \quad 0 < i < \infty.$$

This implies the existence of elements $b_1, b_2, \cdots$ with

$$\text{(10)} \qquad f(a_i \dot{+} c_i) = b_i \dot{+} g(c_i) \quad \text{for} \quad 1 \leqq i < \infty.$$

From (8), (10), and the hypothesis we obtain by 11.3

$$\dot{\sum_{i<\infty}} b_{i+1} \dot{+} \dot{\sum_{i<\infty}} g(c_{i+1}) = \dot{\sum_{i<\infty}} f(a_{i+1} \dot{+} c_{i+1}) \dot{\leqq} b \dot{+} g(c).$$

Hence, by 3.3, 4.37, and 8.13,

$$\dot{\sum_{i<\infty}} b_{i+1} \dot{+} \dot{\sum_{i<\infty}} g(c_{i+1}) \dot{\leqq} b \dot{+} [g(c) \dot{\cap} \dot{\sum_{i<\infty}} f(a_{i+1} \dot{+} c_{i+1})].$$

On the other hand, we have by (9) and 3.10

$$g(c) \dot{\cap} \dot{\sum_{i<\infty}} f(a_{i+1} \dot{+} c_{i+1}) \dot{\leqq} \dot{\sum_{i<\infty}} g(c_{i+1}).$$

The last two inequalities give

$$\dot{\sum_{i<\infty}} b_{i+1} \dotplus \dot{\sum_{i<\infty}} g(c_{i+1}) \leqq b \dotplus \dot{\sum_{i<\infty}} g(c_{i+1}); \tag{11}$$

the existence of the sum on the right side of this formula follows from (8), 11.3, and the hypothesis. Since all elements in $\mathfrak{A}$ are finite by 4.27 and 8.13, we apply 4.19 to (11), and we obtain

$$\dot{\sum_{i<\infty}} b_{i+1} \leqq b.$$

Consequently, there is an element b_0 with

$$b_0 \dotplus \dot{\sum_{i<\infty}} b_{i+1} = \dot{\sum_{i<\infty}} b_i = b. \tag{12}$$

By (8), (10), (12), 11.2, and the hypothesis we have

$$f(a_0 \dotplus c_0) \dotplus \dot{\sum_{i<\infty}} f(a_{i+1} \dotplus c_{i+1}) = [b_0 \dotplus g(c_0)] \dotplus \dot{\sum_{i<\infty}} f(a_{i+1} \dotplus c_{i+1});$$

and by applying 4.19, 4.27, and 8.13 again (or else by 8.16 and 3.9), we get

$$f(a_0 \dotplus c_0) = b_0 \dotplus g(c_0). \tag{13}$$

By (8), (10), (12), and (13), the proof is complete.

We obtain a simple particular case of 11.12 by putting $b = 0$; we can take, in addition, the identity function i for g. Some further consequences of 11.12 are stated in the following

THEOREM 11.13.[4] *Let f and g be partial automorphisms in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$. We then have:*

(i) *if $a, b \;\varepsilon\; A$, $a \mathrel{\dot\leqq} b$, and $f(b) \mathrel{\dot\leqq} g(a)$, then there are elements $c, d, e \;\varepsilon\; A$ such that $a = c \dotplus e$, $b = d \dotplus e$, $g(a) = f(d) \dotplus g(e)$, and $f(b) = g(c) \dotplus f(e)$;*

(ii) *if $a, b, c \;\varepsilon\; A$, $a \mathrel{\dot\leqq} b \mathrel{\dot\leqq} c$, and $g(a) \mathrel{\dot\leqq} f(c)$, then there are elements $d, e \;\varepsilon\; A$ such that $b = d \dotplus e$ and $g(a) \mathrel{\dot\leqq} f(d) \dotplus g(e) \mathrel{\dot\leqq} f(c)$.*

PROOF: (i) By 8.17, 11.4, and 11.11 there is an element c' such that

$$c' \mathrel{\dot\leqq} a \mathrel{\dot\leqq} b \quad \text{and} \quad g(c') = f(b) \mathrel{\dot\leqq} g(a). \tag{1}$$

[4] Particular cases of this theorem and of the related theorem 11.29 can be found in the literature; cf. footnote 17 in §16.

Hence we have for some a_0 and a_1

(2) $$a = a_1 \dotplus c' \quad \text{and} \quad b = a_0 \dotplus a.$$

(1) and (2) imply

$$f(a_0 \dotplus a_1 \dotplus c') = g(c').$$

By applying 11.12 (with $b = 0$) to this formula, we conclude that there exist elements c_0 and c_1 for which

(3) $$c' = c_0 \dotplus c_1, \quad f(a_0 \dotplus c_0) = g(c_0), \quad \text{and} \quad f(a_1 \dotplus c_1) = g(c_1).$$

By putting now

(4) $$c = c_0, \quad d = a_0 \dotplus c_0, \quad \text{and} \quad e = a_1 \dotplus c_1,$$

we see from (1)–(4) and 11.2 that the elements c, d, and e satisfy the conclusion of (i).

(ii) We have for some a_0, a_1, and b'

(5) $$b = a_1 \dotplus a, \quad c = a_0 \dotplus b, \quad \text{and} \quad f(c) = b' \dotplus g(a).$$

Hence

$$f(a_0 \dotplus a_1 \dotplus a) = b' \dotplus g(a).$$

By applying 11.12, we conclude that there are elements b_0, b_1 c_0, and c_1 such that

(6) $$a = c_0 \dotplus c_1, \quad b' = b_0 \dotplus b_1, \quad \text{and} \quad f(a_i \dotplus c_i) = b_i \dotplus g(c_i)$$
$$\text{for} \quad i = 0, 1.$$

We now put

(7) $$d = a_1 \dotplus c_1 \quad \text{and} \quad e = c_0;$$

and from (5)–(7) we easily see that the conclusion of (ii) is satisfied.

In view of 8.18, we can drop the dots over the operation and relation signs in 11.12 and 11.13 if we assume $\mathfrak{A}$ to be multiple-free. Moreover, an analysis of the proof shows that 11.12 and 11.13 thus transformed can be extended to a wider class of algebras; i.e., to all those G.C.A.'s in which every element is finite and in which $a \cup b$ exists for arbitrary elements a and b with a common upper bound. (Only the argument leading to formula (8) in the proof of 11.12 has to be modified in this case.)

DEFINITION 11.14. *Let G be a set of partial automorphisms in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$. Two elements $a, b \varepsilon A$ are called* CONGRUENT UNDER *G, in symbols,*

$$a \underset{G}{\simeq} b,$$

if there is a function $f \varepsilon G$ with $f(a) = b$.

COROLLARY 11.15. *If G is a set of partial automorphisms in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$, we have $0 \underset{G}{\simeq} 0$ unless G is empty; and, for every $a \varepsilon A$, each of the formulas $a \underset{G}{\simeq} 0$ and $0 \underset{G}{\simeq} a$ implies $a = 0$.*

PROOF: by 11.3(i), 11.6(ii), and 11.14.

THEOREM 11.16. *Let G be a set of partial automorphisms in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$. If $0 < n < \infty$, and $a, b, n \cdot a, n \cdot b \varepsilon A$, then $n \cdot a \underset{G}{\simeq} n \cdot b$. implies $a \underset{G}{\simeq} b$.*

PROOF: If f is in G and

$$f(n \cdot a) = n \cdot b,$$

we obtain by 11.2 and 2.34

$$n \cdot f(a) = n \cdot b \quad \text{and} \quad f(a) = b.$$

Hence, by 11.14, the conclusion.

THEOREM 11.17. *If G is a set of partial automorphisms in a G.C.A. $\mathfrak{A}$, then $\underset{G}{\simeq}$ is a finitely and infinitely refining relation both in $\mathfrak{A}$ and $\mathfrak{A}$.*

PROOF: by 6.7, 11.2, 11.4, 11.11, and 11.14.

Some deeper properties of the relation $\underset{G}{\simeq}$ can be derived from 11.12 and 11.13. We obtain, however, a simpler and more interesting formulation of these properties by subjecting the set of transformations G to certain additional assumptions; compare Theorem 11.29 below.

DEFINITION 11.18. *A set G of partial automorphisms in a G.C.A. $\mathfrak{A}$ is called a* QUASI-GROUP, *or simply a* GROUP, *if it satisfies the following conditions:*

(i) $i \varepsilon G$;

(ii) *if $f \varepsilon G$, then $f^{-1} \varepsilon G$;*

(iii) *if $f, g \varepsilon G$, then $fg \varepsilon G$.*

We shall mostly employ the term 'group,' although this is not quite consistent with the usual terminology (since functions in G are not supposed to have a common domain and counter-domain).

Corollary 11.19. *If G is a group of partial automorphisms in a G.C.A. $\mathfrak{A}$, then $\underset{G}{\simeq}$ is an equivalence relation both in $\mathfrak{A}$ and $\dot{\mathfrak{A}}$.*

Proof: by 8.16, 11.14, and 11.18.

The congruence relations $\underset{G}{\simeq}$ discussed so far are not, in general, additive. Therefore, we cannot use them in constructing homomorphic images of G.C.A.'s and apply to them the results obtained in Part I—in the way indicated in 7.13. Hence the idea occurs of extending these relations to additive ones. This can be done in the following way:

Let G be a set of partial automorphisms in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$. We call two elements a and b in $\mathfrak{A}$ equivalent by *(i)* finite, *or (ii)* infinite, decomposition under *G, in symbols,*

$$a \underset{G}{\approx} b, \quad \textit{or} \quad a \underset{G}{\approxeq} b,$$

if there is a number (i) $n < \infty$, or (ii) $n = \infty$, and elements $a_0, a_1, \cdots, b_0, b_1, \cdots \varepsilon A$ such that

$$a = \sum_{i<n} a_i, \quad b = \sum_{i<n} b_i, \quad \textit{and} \quad a_i \underset{G}{\simeq} b_i \quad \textit{for every} \quad i < n.$$

It is clear that the relation $\underset{G}{\approx}$ thus defined is finitely additive, and that the relation $\underset{G}{\approxeq}$ is infinitely additive. Moreover, it can easily be shown that both these relations are finitely and infinitely refining, and that they are equivalence relations in case G is a group; the proof is analogous to that of 7.14.

Assuming now that G is a group, we conclude by 6.10 that $\mathfrak{A}/\underset{G}{\approxeq}$ is a G.C.A., and hence we get a great deal of information concerning $\underset{G}{\approxeq}$. This does not apply, however, to the relation $\underset{G}{\approx}$ (which, on the other hand, plays a more important role than $\underset{G}{\approxeq}$ in certain applications). We can still formally apply 6.3 to this relation and construct the coset algebra $\mathfrak{A}/\underset{G}{\approx}$. However, the sum of an infinite sequence of cosets does not, in general, exist in this algebra, even if the sum of corresponding elements exists in the original algebra $\mathfrak{A}$. Consequently, $\mathfrak{A}/\underset{G}{\approx}$ is not a homomorphic image of $\mathfrak{A}$, and is not, in general, a G.C.A. The situation does not change essentially if we decide to regard G.C.A.'s as algebras with binary addition as the only fundamental operation and to modify Definition 6.3 correspondingly (cf. the remarks which follow 1.36, and the beginning of §13 below). Since the relation $\underset{G}{\approx}$ is finitely additive, $\mathfrak{A}/\underset{G}{\approx}$ proves in this case to be a homomorphic image of the G.C.A. $\mathfrak{A}$; it is, however, by no

means a G.C.A. itself. In consequence, the study of the relation $\underset{G}{\approx}$ is more difficult than that of $\underset{G}{\approxeq}$, and the results are of a more restricted nature.

From the point of view of applications the only important case is the one when the relations $\underset{G}{\approx}$ and $\underset{G}{\approxeq}$ are applied to multiple-free G.C.A.'s. If we are interested only in this particular case, we may, from the beginning, replace ordinary sums which occur in the definitions of these relations by disjunctive sums in the sense of 8.16, and refer subsequent remarks to the algebra $\dot{\mathfrak{A}}$ instead of to $\mathfrak{A}$. However, another procedure turns out to be more suitable here. Instead of introducing new relations into the discussion, we shall single out certain special kinds of automorphism sets, to which we shall refer as FINITELY and INFINITELY ADDITIVE sets of partial automorphisms. It will be seen from Theorem 11.23 that the discussion of the equivalence relations $\underset{G}{\approx}$ and $\underset{G}{\approxeq}$ in multiple-free algebras under arbitrary groups G reduces to the discussion of congruence relations $\underset{H}{\sim}$ under finitely or infinitely additive groups H. This will permit us to establish in a roundabout way some rather interesting properties of equivalence by finite decomposition in multiple-free algebras.

DEFINITION 11.20. *A set G of partial automorphisms in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$ is called* FINITELY ADDITIVE, *or* INFINITELY ADDITIVE, *if it satisfies the following condition:*

Let $n = 2$ (in the case of finite additivity), or $n = \infty$ (in the case of infinite additivity); let $f_0, f_1, \cdots, f_i, \cdots$ be functions in G; and let $a, b, a_0, a_1, \cdots, a_i, \cdots$ be elements in A such that

$$a = \dot{\sum_{i<n}} a_i, \quad b = \dot{\sum_{i<n}} f_i(a_i), \quad \textit{and} \quad a_i \in D(f_i) \quad \textit{for} \quad i < n;$$

then there is a function f in G with

$$D(f) = S(a), \quad C(f) = S(b), \quad \textit{and} \quad f(x) = f_i(x) \quad \textit{for} \quad x \leqq a_i \quad \textit{and} \quad i < n.$$

COROLLARY 11.21. *Every infinitely additive set of partial automorphisms in a G.C.A. $\mathfrak{A}$ is also finitely additive.*

PROOF: by 11.3(i) and 11.20.

THEOREM 11.22. *The set of all partial automorphisms in a G.C.A. $\mathfrak{A}$ is an infinitely additive group.*

PROOF: by 11.6, 11.9, 11.18, and 11.20.

THEOREM 11.23. *For every group G of partial automorphisms in a G.C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$ there is a smallest finitely additive group H of partial automorphisms which includes G. Given any elements a and b in A, we have*

$$a \underset{H}{\simeq} b$$

if, and only if, the following condition is satisfied:

(i) *there is a number $n < \infty$ and there are elements $a_0, a_1, \cdots a_i, \cdots, b_0, b_1, \cdots, b_i, \cdots \varepsilon A$ such that*

$$a = \dot{\sum_{i<n}} a_i, \quad b = \dot{\sum_{i<n}} b_i, \quad \textit{and} \quad a_i \underset{G}{\simeq} b_i \quad \textit{for every} \quad i < n.$$

The theorem remains true if we replace 'finitely additive' by 'infinitely additive,' and '$n < \infty$' by '$n = \infty$'.

PROOF: By 11.21 and 11.22 there are finitely additive groups of partial automorphisms in $\mathfrak{A}$ which include G. From 11.18 it is easily seen that the intersection H of all such groups is again a group; moreover, H is finitely additive since the function f whose existence is required in 11.20 is uniquely determined by 11.9. Thus, H is the smallest finitely additive group which includes G. Now let K be the set of all partial automorphisms f in $\mathfrak{A}$ such that, for any $a, b \;\varepsilon\; A$, the formula $f(a) = b$ implies condition (i) stated in our theorem. By 11.14, K obviously includes G. With the help of 11.18–11.20, we easily show that K is a finitely additive group; to prove that K satisfies 11.18(iii) we apply 2.1, and to show that K is finitely additive we make use of 2.4, 8.16, 11.2, and 11.9. Hence, K includes H; and therefore, by 11.14, the formula

$$a \underset{H}{\simeq} b$$

implies condition (i) for any $a, b \;\varepsilon\; A$. The implication in the opposite direction follows easily from the fact that H is a finitely additive group which includes G; we make use of 11.14 and 11.20. (We can also apply here Theorem 11.24 below, whose proof does not involve 11.23.)—The proof for infinitely additive groups is practically the same.

Theorem 11.23 remains true if we replace 'group' by 'set'; the proof is then much simpler.

THEOREM 11.24. *If G is a finitely, or infinitely, additive set of partial automorphisms in a G.C.A. $\mathfrak{A}$, then $\underset{G}{\simeq}$ is a finitely, or infinitely, additive relation in the G.C.A. $\mathfrak{A}$.*

Proof: by 6.4, 8.16, 8.17, 9.11, 11.2, 11.14, and 11.20.

Theorem 11.25. *If G is an infinitely additive group of partial automorphisms in a G.C.A. $\mathfrak{A}$, then $\underset{G}{\simeq}$ is an infinitely additive and infinitely refining equivalence relation in the G.C.A. $\dot{\mathfrak{A}}$, and $\dot{\mathfrak{A}}/\underset{G}{\simeq}$ is a G.C.A.*

Proof: by 6.10, 8.17, 11.17, 11.19, and 11.24.

By this theorem, all the arithmetical results of Part I hold in every coset algebra $\dot{\mathfrak{A}}/\underset{G}{\simeq}$ where G is any infinitely additive group of partial automorphisms in $\mathfrak{A}$.

As regards the case when the group G is finitely additive, we can repeat the remarks previously made in connection with the relation $\underset{G}{\approx}$. By applying 6.3, we can construct the algebra $\dot{\mathfrak{A}}/\underset{G}{\simeq}$ in this case as well. This algebra, however, will not be a G.C.A., and it will be a homomorphic image of the G.C.A. $\dot{\mathfrak{A}}$ only under binary addition. Nevertheless, we can extend to such algebras $\dot{\mathfrak{A}}/\underset{G}{\simeq}$ a number of arithmetical theorems stated in Part I; and among them some results which are not of an elementary nature. All these results can be derived from a few simple postulates. We introduce a special term to denote algebras satisfying these postulates, since we shall have to refer to them several times in our further discussion.

Definition 11.26. *An algebraic system $\mathfrak{A} = \langle A, +, \sum \rangle$ is called a* REFINEMENT ALGEBRA, *for abbreviation, an R.A., if it satisfies the following five postulates:*

I. *If $a, b, a + b \in A$, then $a + b = b + a$.*

II. *If $a, b, c, a + b, (a + b) + c \in A$, then $b + c \in A$ and $(a + b) + c = a + (b + c)$.*

III. *There is an element $z \in A$ such that $a + z = a$ for every $a \in A$, and $a + b = z$ implies $b = z$ for every $a, b \in A$.*

IV. *If $a_1, a_2, b_1, b_2, a_1 + a_2 \in A$ and $a_1 + a_2 = b_1 + b_2$, then there are elements $c_1, c_2, c_3, c_4 \in A$ such that*

$$a_1 = c_1 + c_2, \quad a_2 = c_3 + c_4, \quad b_1 = c_1 + c_3, \quad \textit{and} \quad b_2 = c_2 + c_4.$$

V. *If $a_1, a_2, b, c, a_1 + a_2, (a_1 + a_2) + c \in A$ and $(a_1 + a_2) + c = b + c$, then there are elements $b_1, b_2, c_1, c_2 \in A$ such that*

$$b = b_1 + b_2, \quad c = c_1 + c_2, \quad a_1 + c_1 = b_1 + c_1, \quad \textit{and}$$

$$a_2 + c_2 = b_2 + c_2.$$

The term 'refinement algebra' has been chosen in view of the fact that two of the postulates listed above, namely, IV and V, have the character of refinement theorems; in fact, they differ from 2.3 and 2.18 (with $n = 2$) only by the presence of existential assumptions. It is easy to formulate one postulate which can equivalently replace both IV and V.

As is seen from Definition 11.26, infinite addition $\sum$ is not involved in the postulate system at all; hence, the term 'refinement algebra' can be more properly applied to algebras with one (binary) operation, discussed in Part III of this work.

Theorem 11.27. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A. Then*

(i) $\mathfrak{A}$ *is an R.A.;*

(ii) *if G is a finitely additive group of partial automorphisms in* $\mathfrak{A}$, *then* $\mathfrak{A}/\widetilde{G}$ *is also an R.A.*

Proof: The first part of the theorem is obvious; compare 11.26 with 5.8, 5.9, 5.4 and 5.19, 5.23 for $n = p = 2$ (in view of 5.6), and 2.18 for $n = 2$. As regards the second part, Postulates 11.26.I–IV can be derived directly for the algebra $\mathfrak{A}/\widetilde{G}$ from the corresponding theorems of the arithmetic of G.C.A.'s just listed; for, whenever any one of these postulates holds in a given algebra $\mathfrak{A}$, it also holds in every coset algebra $\mathfrak{A}/R$ generated by a finitely additive and finitely refining equivalence relation R. (Here we apply 6.3, 6.4, 6.7, 11.17, 11.19, and 11.24.) Postulate 11.26.V does not have this property, and hence it cannot be obtained directly from the corresponding arithmetical theorem (2.18); it follows immediately, however, from 11.12 (with $n = 2$) and 11.14.

In our further discussion we shall come across certain interesting examples of R.A.'s obtained with the help of 11.27(ii).

Theorem 11.28. *Let* $A = \langle A, +, \sum \rangle$ *be an R.A. We extend to* $\mathfrak{A}$ *all definitions stated in Part* I*; instead of* 1.3, *however, we assume a recursive definition of the sum of a finite sequence:*

$$\sum_{i<0} a_i = 0, \quad \text{and} \quad \sum_{i<n+1} a_i = \sum_{i<n} a_i + a_n \quad \text{for every} \quad n < \infty$$

(in case $\sum_{i<n} a_i, \sum_{i<n} a_i + a_n \;\varepsilon\; A$*).*

Then the following theorems, when restricted to finite sequences, sums, and multiples, and when provided with existential assumptions required in G.C.A.'s, hold in the algebra $\mathfrak{A}$*:*

(i) *all theorems of* §1 (*except* 1.10, 1.29, 1.36, *and* 1.37)*;*

(ii) *Theorems* 2.1–2.5, 2.16–2.18, 2.27 *with* $n = 1$, 2.28, *and* 2.30 *with* $n = p = 1$ *of* §2*;*

(iii) *Theorems* 3.3–3.5, 3.7, 3.8, 3.10–3.18, 3.23, *and* 3.31–3.34 *of* §3*;*

(iv) *Theorems* 4.2, 4.3 *without* (iii), 4.4, 4.6–4.9, 4.11, 4.13–4.15, 4.18, 4.27–4.37, *and* 4.39–4.48 *of* §4.

PROOF: The proofs of most theorems of §1 either remain unchanged or (as in the case of 1.7 and 1.11) can be replaced by simple inductive proofs based upon 11.26.I–III. By taking for b in 11.26.V the zero element z of 11.26.III, we easily obtain 1.30 and 1.31, and also, by induction, 1.46 and 1.47. In §2, Theorem 2.3 coincides with 11.26.IV; 2.1 follows from 2.3 by induction, and 2.2, 2.4, and 2.5 are simple consequences of 2.1 and 2.3. Furthermore, 2.16 and 2.17 can be obtained by induction from 11.26.I–III. 2.18 obviously holds for $n = 1$, it follows directly from 11.26.V for $n = 2$, and the result can be extended by induction to an arbitrary $n < \infty$. 2.27 with $n = 1$ is an immediate corollary of 2.18 (for $n = 2$), and it implies in turn 2.30 with $n = p = 1$. The original proof of 2.28 shows that this theorem for $n = p = 2$ can be derived from 2.3 and 2.27 (with $n = 1$); an extension by induction to arbitrary $n < \infty$ and $p < \infty$ presents no difficulty. The proofs of theorems in §§3 and 4 remain practically unchanged; in 3.31 and 4.35, the second variants of the proofs are used. (In view of the elementary nature of the arguments, the lack of the closure postulate 1.1.I does not cause any essential difficulties.)

As regards arithmetical theorems which have not been listed in 11.28, in several cases examples are known which show that these theorems may fail in R.A.'s, and in particular, in coset algebras $\mathfrak{A}/\underset{G}{\sim}$ of Theorem 11.27; this applies, e.g., to Theorems 2.6–2.9 and 2.19. In other cases the problem remains open. It is not known, for instance, whether 3.29 holds in refinement algebras. On the other hand, the distributive law 3.30 for $n < \infty$ can be derived from the dual distributive law, 3.32, which has been listed in 11.28. It seems, however, that in this case the hypothesis of 3.30 must be provided with stronger existential assumptions; even the assumption of the existence of all bounds which are directly involved in the conclusion of 3.30 does not seem to suffice.

Notice that the fundamental theorems on finite elements, 4.16 and 4.19, have not been listed in 11.28. It would probably be more convenient to define a finite element (of an R.A.) as an element c which satisfies 4.19(ii); under this definition almost all theorems on finite elements of §4 would prove to hold in R.A.'s.

A detailed study of R.A.'s will not be undertaken here. The following rather interesting property of these algebras can easily be established:

Let $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a finitely closed R.A., and let* c *be an element of* A. *We agree to call two elements* a *and* b EQUIVALENT UNDER c $a \underset{c}{\equiv} b$, *if* $a + c = b + c$. *Then* $\underset{c}{\equiv}$ *is a finitely additive and finitely refining equivalence relation in* $\mathfrak{A}$, *and* $\mathfrak{A}/\underset{c}{\equiv}$ *is again an R.A.*

By modifying the definition of $\underset{c}{\equiv}$, we can extend this result to arbitrary R.A.'s.

Some further arithmetical results applying to coset algebras $\dot{\mathfrak{A}}/\underset{G}{\simeq}$ of 11.27 can be obtained by specializing the assumptions which concern $\mathfrak{A}$ or G; these results will be discussed briefly in §16.

From 11.25 and 11.27(ii) we can derive various deeper properties of the congruence relation $\underset{G}{\simeq}$ under a finitely or infinitely additive group G of partial automorphisms. Thus, for instance, if G is infinitely additive, we can apply 7.13 by taking the algebra $\dot{\mathfrak{A}}$ for $\mathfrak{A}$ and the relation $\underset{G}{\simeq}$ for R. The following theorem shows that some of the conclusions thus obtained—in fact, (i), (ii), (iii) with $n < \infty$, and (vi) of 7.13—still hold in the case when G is finitely additive.

THEOREM 11.29. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A., and let* G *be a finitely additive group of partial automorphisms in* $\mathfrak{A}$. *We then have for any elements* $a, a', a_0, a_1, \cdots, b, b', c, c' \in A$:

(i) *if* $a \dot{\leq} b$, $a' \dot{\leq} b'$, $a \underset{G}{\simeq} b'$, *and* $b \underset{G}{\simeq} a'$, *then* $a \underset{G}{\simeq} a'$ *and* $b \underset{G}{\simeq} b'$;

(ii) *if* $a \dot{\leq} b \dot{\leq} c$ *and* $a \underset{G}{\simeq} c$, *then* $a \underset{G}{\simeq} b \underset{G}{\simeq} c$;

(iii) *if* $n < \infty$ *and* $\dot{\sum}_{i<n} a_i$, $\dot{\sum}_{i<n} a_i \dotplus b \in A$, *then*

$$\dot{\sum_{i<n}} a_i \dotplus b \underset{G}{\simeq} b$$

implies that $a_i \dotplus b \underset{G}{\simeq} b$ *for every* $i < n$, *and conversely;*

(iv) *if* $a \dot{\leq} b \dot{\leq} c$, $a' \dot{\leq} c'$, $a \underset{G}{\simeq} a'$, *and* $c \underset{G}{\simeq} c'$, *then there is an element* $b \in A$ *such that* $a' \dot{\leq} b' \dot{\leq} c'$ *and* $b \underset{G}{\simeq} b'$.

PROOF: entirely analogous to that of 7.13; we take into account 8.17, and apply the properties of $\widetilde{}_G$ and $\mathfrak{A}/\widetilde{}_G$ stated in 11.17, 11.19, 11.24, 11.27, and 11.28.

We can also derive the conclusions of 11.29 directly from 11.13, without the help of 11.27 and 11.28. For instance, by applying 11.14 to the hypothesis of 11.29(i), we obtain two partial automorphisms f and g in G such that

$$f(b) = a' \quad \text{and} \quad g(a) = b'.$$

Thus, the hypothesis of 11.13(i) is satisfied; and we have for some c, d, and e

$$a = c \dotplus e, \quad a' = g(c) \dotplus f(e), \quad b = d \dotplus e, \quad \text{and} \quad b' = f(d) \dotplus g(e).$$

Hence, by means of 6.4, 11.14, and 11.24, we obtain the conclusion of 11.29(i). In an analogous way 11.29(iv) can be derived from 11.13(ii); while 11.29(ii) and 11.29(iii) are simple consequences of 11.19 and 11.29(i). Such a direct proof of 11.29 is even more advantageous, for it shows that most conclusions of the theorem can be obtained under a weaker hypothesis. In fact, 11.29(i),(iv), can be derived under the assumption that G is an arbitrary finitely additive set (and not necessarily a group) of automorphisms; 11.29(ii) and the first half of 11.29(iii) require the additional assumption that G contains the identity automorphism; and the fact that G is a group is used only in deriving the second part of 11.29(iii).

We now want to discuss briefly a special congruence relation —that of HOMOGENEITY.

DEFINITION 11.30. *Let $\mathfrak{A} = \langle A, +, \sum\rangle$ be a G.C.A. Two elements $a, b \,\varepsilon\, A$ are called* HOMOGENEOUS, *in symbols,*

$$a \sim b,$$

if they are congruent under the set of all partial automorphisms in $\mathfrak{A}$.

THEOREM 11.31. *Let $\mathfrak{A} = \langle A, +, \sum\rangle$ be a G.C.A. For any $a, b \,\varepsilon\, A$ the formulas*

$$a \sim b \quad \textit{and} \quad S(a) \cong S(b)$$

are equivalent.

PROOF: by 6.1, 6.13, 9.11, 11.3, 11.4, 11.6(iii), 11.14, and 11.30, without any difficulty.

THEOREM 11.32. *The relation* $\sim$ *in an arbitrary G.C.A.* $\mathfrak{A}$ *is an infinitely refining equivalence relation in* $\mathfrak{A}$*; moreover, it is an infinitely additive and infinitely refining equivalence relation in the G.C.A.* $\mathring{\mathfrak{A}}$, *and hence* $\mathring{\mathfrak{A}}/\sim$ *is a G.C.A.*

PROOF: by 11.17, 11.19, 11.22, 11.24, 11.25, and 11.31.

It should be noticed that the relations of homogeneity $\sim$ in $\mathfrak{A}$ and $\mathring{\sim}$ in $\mathring{\mathfrak{A}}$ do not necessarily coincide. However, Theorem 11.32 in its part referring to $\mathring{\mathfrak{A}}$ remains true if we replace in it '$\sim$' by '$\mathring{\sim}$'; this is clearly seen from 8.17 and 8.18.

THEOREM 11.33. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A. If* $n \leqq \infty$ *and* $a, b, n \cdot a, n \cdot b \in A$, *then* $a \sim b$ *implies* $n \cdot a \sim n \cdot b$*; if* $0 < n < \infty$, *then the converse implication also holds.*

PROOF: If f is a partial automorphism and $f(a) = b$, then by 11.7 there is a partial automorphism g with $g(n \cdot a) = n \cdot b$. Hence, by 11.14 and 11.30, we obtain the first implication at once; the converse follows from 11.16 and 11.30.

COROLLARY 11.34. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A. If* a, b, $\infty \cdot a$, $\infty \cdot b \in A$, *then* $a \sim b$ *implies* $I(a) \cong I(b)$*; if the elements* a *and* b *are idem-multiple, then the converse also holds.*

PROOF: by 9.13, 11.31, and 11.33.

The last two theorems in this section refer to the algebra of ideals which was discussed in §10.

LEMMA 11.35. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A.; for every partial automorphism* f *in* $\mathfrak{A}$ *and for every ideal* X *in* $\mathfrak{A}$ *which is included in* $D(f)$, *let* $f^*(X)$ *be the set of all elements* $f(x)$ *with* $x \in X$*; and let* G *be the class of all functions* f^* *thus defined. Then*

(i) *G is a group of partial automorphisms in the ideal algebra* $\mathfrak{J}(\mathfrak{A})$*;*

(ii) *G is finitely additive in case* $\mathfrak{A}$ *is a finitely closed G.C.A., and is infinitely additive in case* $\mathfrak{A}$ *is a C.A.;*

(iii) *the formulas*

$$B \underset{G}{\sim} C \quad \text{and} \quad B \cong C$$

are equivalent for all ideals B *and* C *in* $\mathfrak{A}$.

PROOF: (i) The reasoning is obvious and requires no special idea. We use 6.1, 9.1, 9.2, 9.12, 10.1–10.4, 11.1–11.3, and 11.6. With the help of these definitions and theorems we easily establish the following facts:

(1) if i is the identity function in $\mathfrak{A}$, then i^* is the identity function in $\mathfrak{I}(\mathfrak{A})$;
furthermore, f and g being any partial automorphisms in $\mathfrak{A}$, we have
(2) f^* is biunique and $(f^*)^{-1} = (f^{-1})^*$;
(3) $f^*g^* = (fg)^*$;
(4) $D(f^*)$ and $C(f^*)$ are semi-ideals in $\mathfrak{I}(\mathfrak{A})$;
(5) the formulas $B = C + D$ and $f^*(B) = f^*(C) + f^*(D)$ are equivalent for any ideals B, C, and D in $\mathfrak{A}$.
Hence the conclusion follows by 11.1, 11.6, 11.18, and the definition of G.

(ii) Consider partial automorphisms f_i in $\mathfrak{A}$ and ideals B, C, and A_i such that

(6) $$B = \dot{\sum_{i<n}} A_i, \quad C = \dot{\sum_{i<n}} f^*(A_i), \quad \text{and} \quad A_i \,\varepsilon\, D(f_i^*) \quad \text{for} \quad i < n,$$

where $n = 2$ in case $\mathfrak{A}$ is a finitely closed G.C.A., and $n = \infty$ in case $\mathfrak{A}$ is a C.A.
The functions

$$g_i = f_i^{[A_i]}$$

are again partial automorphisms by 11.6(iii); and we have by (6), 8.16, and 10.3

$$D(g_i) \cap D(g_j) = A_i \cap A_j = I(0)$$

and

$$C(g_i) \cap C(g_j) = f^*(A_i) \cap f^*(A_j) = I(0)$$

for all i and j, $i < j < n$. Hence, by 11.10, there is a partial automorphism f in $\mathfrak{A}$ with

$$D(f) = \dot{\sum_{i<n}} A_i, \quad C(f) = \dot{\sum_{i<n}} f^*(A_i), \quad \text{and} \quad f(x) = g_i(x) = f_i(x)$$

for $x \,\varepsilon\, A_i$.

We easily conclude by 9.11 and 10.4 that

(7) $$D(f^*) = S(B) \quad \text{and} \quad C(f^*) = S(C);$$

(8) $$f^*(X) = f_i^*(X) \quad \text{for} \quad X \leqq A_i \quad (X \text{ an ideal in } \mathfrak{A}).$$

(The symbols '$\leqq$', '$S(B)$', and '$S(C)$' in (7) and (8) apply to the algebra $\mathfrak{I}(\mathfrak{A})$.) Thus, (6) implies the existence of a function f which

satisfies (7) and (8); hence, according to 11.20, G is finitely, or infinitely, additive in $\mathfrak{J}(\mathfrak{A})$.

Finally, (iii) follows easily from 6.1, 6.13, 9.1, 11.1, and 11.6(iii). This completes the proof.

THEOREM 11.36. *Let $\mathfrak{A}$ be a G.C.A.*

(i) *The relation of isomorphism $\cong$ between ideals in $\mathfrak{A}$ is an infinitely refining equivalence relation both in the ideal algebra $\mathfrak{J}(\mathfrak{A})$ and in the correlated disjunctive algebra $\dot{\mathfrak{J}}(\mathfrak{A})$.*

(ii) *If $\mathfrak{A}$ is finitely closed, then the relation $\cong$ is finitely additive in $\dot{\mathfrak{J}}(\mathfrak{A})$, and the algebra $\dot{\mathfrak{J}}(\mathfrak{A})/\cong$ is an R.A.*

(iii) *If $\mathfrak{A}$ is a C.A., then the relation $\cong$ is infinitely additive in $\dot{\mathfrak{J}}(\mathfrak{A})$, and $\dot{\mathfrak{J}}(\mathfrak{A})/\cong$ is a G.C.A.*

PROOF: by 10.3, 11.17, 11.19, 11.24, 11.27, and 11.35.

Notice that, by 11.14, 11.30, and 11.35, the isomorphism of two ideals implies their homogeneity in the ideal algebra. The converse in general fails (it probably holds in idem-multiple algebras). Thus, in case $\mathfrak{A}$ is a C.A., $\dot{\mathfrak{J}}(\mathfrak{A})/\sim$ and $\dot{\mathfrak{J}}(\mathfrak{A})/\cong$ are both G.C.A.'s (by 11.32 and 11.36), but they by no means coincide.

§ 12. ISOMORPHISM TYPES OF CARDINAL ALGEBRAS

Algebras of ISOMORPHISM TYPES under CARDINAL MULTIPLICATION, which will be studied in this section, are related to the ideal algebras constructed in §10, and especially to their coset algebras under the relation of isomorphism discussed at the end of §11.

When speaking here of an algebra $\mathfrak{A}$, we shall always understand an algebra with a zero element.

DEFINITION 12.1. *By the* ISOMORPHISM TYPE *(or simply the* TYPE*)* $\tau(\mathfrak{A})$ OF AN ALGEBRA $\mathfrak{A} = \langle A, +, \sum \rangle$ *we understand the class of all algebras isomorphic with* $\mathfrak{A}$*;* α *is called an* ISOMORPHISM TYPE *if there is an algebra* $\mathfrak{A}$ *for which* $\alpha = \tau(\mathfrak{A})$.

The notion of an isomorphism type thus defined may appear antinomial; and this applies even more to various classes of isomorphism types which will be involved in our further discussion, e.g., to the class of all isomorphism types of C.A.'s. The doubts which may arise in this connection cannot be entirely dispelled without a detailed discussion of the set-theoretical foundations of our work. This is, however, a task which we have no intention of undertaking here.[5] One who is acquainted with modern investigations into the foundations of set theory will certainly agree that a satisfactory solution of the problem is not only possible but can be obtained in many different ways; the difficulties can be removed, e.g., by a restriction to algebras and sets of not too high a power, or by a strict differentiation between sets and classes, and possibly some 'superclasses.' (This differentiation, by the way, is not carried through consistently in the present work.)

On the other hand, we could avoid any appearance of antinomial constructions by dispensing with notions like that of isomorphism type (or, e.g., cardinal number) altogether, or at least by refraining from any discussion of sets of such types and numbers. This would not impoverish the mathematical content of our results, but would prevent us from putting some of them in a simple and suggestive form.

[5] The reader interested in the foundations of set theory may be referred to the most recent work in this domain—Bernays [1]: he will find there a bibliography of the subject.

We shall be interested here mainly in types of C.A.'s and G.C.A.'s; however, we shall formulate explicitly certain elementary theorems applying to arbitrary isomorphism types. It will be seen in the appendix to this work that some results obtained for types of C.A.'s and G.C.A.'s in the present section can be considerably generalized.

DEFINITION 12.2. (i) *By the* CARDINAL (*or* DIRECT) PRODUCT $\prod_{i \varepsilon I} \alpha_i$ OF ISOMORPHISM TYPES α_i *correlated with elements i of an arbitrary set I we understand the unique type α which satisfies the following condition: if $\mathfrak{A}_i$ are any algebras such that*

$$\alpha_i = \tau(\mathfrak{A}_i) \quad \textit{for every} \quad i \varepsilon I,$$

then

$$\alpha = \tau\,(\prod_{i \varepsilon I} \mathfrak{A}_i).$$

(ii) *If I consists of two numbers,* 0 *and* 1, *and if $\alpha_0 = \beta$ and $\alpha_1 = \gamma$, we put*

$$\alpha = \beta \times \gamma.$$

(iii) *Similarly we define the* STRONG *and the* WEAK CARDINAL PRODUCT OF ISOMORPHISM TYPES,

$$\prod_{i \varepsilon I}{}_s\, \alpha_i \quad \textit{and} \quad \prod_{i \varepsilon I}{}_w\, \alpha_i\,.$$

Compare here Definition 6.11.

We shall pay but little attention to strong and weak cardinal products of types—for the reasons which have been explained in connection with 6.12.

THEOREM 12.3. *Let α_i be isomorphism types correlated with elements $i \varepsilon I$. Then:*

(i) *$\prod_{i \varepsilon I} \alpha_i$ always exists and is again an isomorphism type; and if I consists of one element j, we have*

$$\prod_{i \varepsilon I} \alpha_i = \alpha_j\,.$$

(ii) *The same applies to $\prod_{i \varepsilon I}{}_s\, \alpha_i$ and $\prod_{i \varepsilon I}{}_w\, \alpha_i$; moreover, if I is finite, or at most denumerable, we have*

$$\prod_{i \varepsilon I} \alpha_i = \prod_{i \varepsilon I}{}_w\, \alpha_i\,, \quad \textit{or} \quad \prod_{i \varepsilon I} \alpha_i = \prod_{i \varepsilon I}{}_s\, \alpha_i\,,$$

respectively (so that, in particular, $\alpha \times \beta$ is also the strong and the weak product of α and β).

PROOF: by 6.1, 6.11, 12.1, and 12.2; we notice that two products of pairwise isomorphic algebras are always isomorphic.

In addition to the general closure postulate 12.3, cardinal products satisfy the most general commutative and associative laws:

THEOREM 12.4. *Let I and J be any two sets, and let f be a function which maps I onto J in a one-to-one way. If, for every $i \varepsilon I$, α_i is an isomorphism type and $\alpha_i = \beta_{f(i)}$, then*

$$\prod_{i \varepsilon I} \alpha_i = \prod_{j \varepsilon J} \beta_j .$$

Similarly for strong and weak cardinal products.

PROOF: We notice that 6.11 implies a formula for products of algebras, which is analogous to that given in the conclusion, but in which '$=$' is replaced by '$\cong$'; hence, by 12.1–12.3, the conclusion.

THEOREM 12.5. *Let a set J_i be correlated with every element $i \varepsilon I$; let K be the set of all ordered couples $\langle i, j \rangle$ with $i \varepsilon I$ and $j \varepsilon J_i$. If, for every $\langle i, j \rangle$ in K, $\alpha_{i,j}$ is an isomorphism type, then*

$$\prod_{\langle i,j \rangle \varepsilon K} \alpha_{i,j} = \prod_{i \varepsilon I} \prod_{j \varepsilon J_i} \alpha_{i,j} .$$

Similarly for strong and weak cardinal products.

PROOF: analogous to that of 12.4.

We shall concern ourselves with algebras $\langle \mathrm{T}, \times, \prod \rangle$ where T is a class of isomorphism types; the symbol '$\prod$' occurring in the denotation of such an algebra is understood to refer to products of infinite sequences only (cf. the remarks which follow 8.7), but otherwise it will be used in the unrestricted sense. According to 5.2, all the arithmetical notions introduced in Part I are automatically extended to algebras $\langle \mathrm{T}, \times, \prod \rangle$; we use here, however, a multiplicative terminology, and, in case T is the class of all isomorphism types, we omit the reference to the algebra concerned. Thus, we say that a type α is a FACTOR of a type β (and not that $\alpha \leqq \beta$) if there is a type γ for which $\alpha \times \gamma = \beta$. Similarly, as is seen from Theorem 12.7 below, the zero element of the algebra of all isomorphism types is denoted by '1'.

DEFINITION 12.6. *By the* UNIT TYPE, 1, *we understand the common isomorphism type of all algebras* $\mathfrak{A} = \langle A, +, \sum \rangle$ *where* A *consists of one element* z *and where*

$$z + z = \sum_{i<\infty} z = z.$$

THEOREM 12.7. (i) *The isomorphism type* 1 *exists, and we have* $\alpha \times 1 = 1 \times \alpha = \alpha$ *for every type* α.

(ii) α_i *being isomorphism types correlated with elements* $i \varepsilon I$, *we have*

$$\prod_{i \varepsilon I} \alpha_i = 1$$

if, and only if, $\alpha_i = 1$ *for every* $i \varepsilon I$. *Similarly for strong and weak cardinal products.*

PROOF: by 6.1, 6.11, 12.1, 12.2, and 12.8.

By extending Definition 4.38 to the algebra of isomorphism types, we obtain

DEFINITION 12.8. *An isomorphism type* α *is called* INDECOMPOSABLE *if* $\alpha \neq 1$ *and if, for any types* ζ *and* η, $\alpha = \zeta \times \eta$ *implies that* $\zeta = 1$ *or* $\eta = 1$.

Turning now to types of C.A.'s and G.C.A.'s, we have first

THEOREM 12.9. *Let* α_i *be isomorphism types correlated with elements* $i \varepsilon I$.

(i) *In order that* $\prod_{i \varepsilon I} \alpha_i$ *be the type of a C.A., or of a G.C.A., or of a finitely closed G.C.A., it is necessary and sufficient that, for every* $i \varepsilon I$, α_i *be the type of a C.A., or of a G.C.A., or of a finitely closed G.C.A.*

(ii) *The same applies to* $\prod^{s}_{i \varepsilon I} \alpha_i$ *and—in the case of G.C.A.'s and of finitely closed G.C.A.'s—also to* $\prod^{w}_{i \varepsilon I} \alpha_i$. *However,* $\prod^{w}_{i \varepsilon I} \alpha_i$ *is never the type of a C.A., unless there are only finitely many elements* $i \varepsilon I$ *for which* $\alpha_i \neq 1$.

PROOF: by 6.11, 6.12, 12.1–12.3, and 12.6.

THEOREM 12.10. 1 *is the type of a C.A.*

PROOF: by 1.1, 12.6, and 12.7(i).

THEOREM 12.11. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A., and let* α_i *be isomorphism types correlated with elements* i *of a set* I. *In order that*

$$\tau(\mathfrak{A}) = \prod_{i \varepsilon I} \alpha_i$$

it is necessary and sufficient that it be possible to correlate an ideal A_i *in* $\mathfrak{A}$ *with every element* $i \varepsilon I$, *so as to satisfy the following conditions:*

(i) $A = \bigcup_{i \varepsilon I} A_i$, *and* $A_i \cap A_j = I(0)$ *for all* $i, j \varepsilon I$ *with* $i \neq j$;

(ii) *if* $n \leqq \infty$, $a_k \varepsilon A_{i_k}$ *and* $i_k \varepsilon I$ *for every* $k < n$, *and* $i_k \neq i_l$ *for all* k *and* l *with* $k < l < n$, *then* $\sum_{k<n} a_k \varepsilon A$;

(iii) $\tau(\langle A_i, +, \sum \rangle) = \alpha_i$.

PROOF: Assume that ideals A_i which satisfy (i)–(iii) are given, and let

(1) $$\mathfrak{A}_i = \langle A_i, +, \textstyle\sum \rangle.$$

We form the cardinal product

(2) $$\prod_{i \varepsilon I} \mathfrak{A}_i = \langle F, +, \textstyle\sum \rangle.$$

By 6.12, 6.13, and 9.18,

(3) $$\prod_{i \varepsilon I} \mathfrak{A}_i \quad \text{is a G.C.A.}$$

By 6.11, the set F in (2) consists of functions f with

$$D(f) = I, \quad \text{and} \quad f(i) \varepsilon A_i \quad \text{for} \quad i \varepsilon I.$$

Moreover, there are at most denumerably many elements $i \varepsilon I$ for which $f(i) \neq 0$. We arrange all such elements i in a sequence i_0, $i_1, \cdots, i_k, \cdots$ with $k < n \leqq \infty$ (and without repetitions). By (ii), the sum of corresponding function values $f(i_k)$ exists, and we put

$$a = G(f) = \sum_{k<n} f(i_k).$$

The domain of the function G thus defined is F; from (i), 6.11, and 10.7 we see that its counter-domain is A. Thus,

(4) $$D(G) = F \quad \text{and} \quad C(G) = A.$$

Consider now any three functious f, g, and h in F; and let

(5) $$a = G(f), \quad b = G(g), \quad \text{and} \quad c = G(h)$$

be the correlated elements of A. From 6.11 and the definition of G we easily conclude that the formula

(6) $$f = g + h$$

implies

(7) $$a = b + c.$$

Now, conversely, assume (7) to hold. We arrange all those elements $i \in I$ for which $f(i)$, $g(i)$, or $h(i)$ is different from 0 in one sequence $i_0, i_1, \cdots, i_k, \cdots$ with $k < n \leqq \infty$ (and without repetitions). We then obtain by (5)

$$a = \sum_{k<n} f(i_k), \quad b = \sum_{k<n} g(i_k), \quad c = \sum_{k<n} h(i_k),$$

and hence

$$a = \sum_{k<n} f(i_k) = \sum_{k<n} [g(i_k) + h(i_k)]. \tag{8}$$

We have

$$f(i_k), \quad g(i_k), \quad h(i_k) \in A_{i_k}; \tag{9}$$

the sets A_{i_k} being ideals by hypothesis, we get further by 9.3

$$g(i_k) + h(i_k) \in A_{i_k} \quad \text{for} \quad k < n. \tag{10}$$

Moreover, the ideals A_{i_k} are by (i) pairwise disjoint. Hence and from (8)–(10) we easily conclude by means of 3.11 and 10.6 that

$$f(i_k) = g(i_k) + h(i_k) \quad \text{for} \quad k < n.$$

Consequently,

$$f(i) = g(i) + h(i) \quad \text{for every} \quad i \in I,$$

and, in view of 6.11, we arrive at (6). Thus, the formulas

$$f = g + h \quad \text{and} \quad G(f) = G(g) + G(h)$$

are equivalent for any functions $f, g, h \in F$. By applying 6.2 we infer hence from (2)–(4) that G maps $\prod_{i \in I} \mathfrak{A}_i$ isomorphically onto $\mathfrak{A}$, and that in consequence

$$\mathfrak{A} \cong \prod_{i \in I} \mathfrak{A}_i .$$

Hence, by (iii), (1), and 12.1–12.3,

$$\tau(\mathfrak{A}) = \prod_{i \in I} \alpha_i . \tag{11}$$

This completes the proof of our theorem in one direction.

Now assume (11) to hold. By 12.1, 12.2, and 12.10, there exist G.C.A.'s

$$\mathfrak{A}_i = \langle A'_i, +_i, \textstyle\sum_i \rangle$$

such that

$$\mathfrak{A} \cong \prod_{i \varepsilon I} \mathfrak{A}_i, \quad \text{and} \quad \tau(\mathfrak{A}_i) = \alpha_i \quad \text{for every} \quad i \varepsilon I.$$

Let H be the function which maps $\mathfrak{A}$ isomorphically onto $\prod_{i \varepsilon I} \mathfrak{A}_i$. Thus, a being any element in A, $H(a)$ is a function f with

$$D(f) = I, \quad \text{and} \quad f(i) \varepsilon A'_i \quad \text{for} \quad i \varepsilon I;$$

and the set of all elements $i \varepsilon I$ for which

$$f(i) \neq 0_i$$

is at most denumerable. For any given element $i \varepsilon I$, let A_i be the set of all elements $a \varepsilon A$ such that

$$f(j) = 0_j \quad \text{for} \quad f = H(x) \quad \text{and for every} \quad j \varepsilon I \quad \text{with} \quad j \neq i.$$

The proof that the sets A_i thus defined are ideals in $\mathfrak{A}$ and that they satisfy conditions (i)–(iii) of our theorem is based upon 6.1, 6.11, 9.1, 10.6, and 10.7, and presents no difficulty. Thus, Theorem 12.11 holds in both directions.

COROLLARY 12.12. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a G.C.A., and let* $\alpha_0, \alpha_1, \cdots, \alpha_i, \cdots$ *with* $i < n \leqq \infty$ *be isomorphism types. In order that*

$$\tau(\mathfrak{A}) = \prod_{i<n} \alpha_i$$

it is necessary and sufficient that it be possible to correlate an ideal A_i *in* $\mathfrak{A}$ *with every* $i < n$, *so as to satisfy the following conditions:*

(i) $A = \sum_{i<n} A_i$, *and* $A_i \cap A_j = I(0)$ *for all* i, j *with* $i < j < n$;

(ii) *if* $a_i \varepsilon A_i$ *for every* $i < n$, *then* $\sum_{i<n} a_i \varepsilon A$;

(iii) $\tau(\langle A_i, +, \sum \rangle) = \alpha_i$ *for every* $i < n$.

PROOF: by 9.2, 10.8, and 12.11.

THEOREM 12.13. *Theorem* 12.11 *applies to weak cardinal products if we replace in it condition* (ii) *by the following one:*

(ii′) *if* $n \leqq \infty$, $a_k \varepsilon A_{i_k}$ *and* $i_k \varepsilon I$ *for every* $k < n$, *and* $i_k \neq i_l$ *for all* k *and* l *with* $k < l < n$, *then* $\sum_{k<n} a_k$ *is in* A *if, and only if,* $n < \infty$.

PROOF: entirely analogous to that of 12.11.

We could also transform 12.11 so as to make it apply to strong cardinal products; this would require, however, a more radical change in the formulation of the theorem.

The value of Theorems 12.11–12.13 lies in the fact that they reduce the study of cardinal products of algebras and isomorphism types to the discussion of relations between certain subalgebras (ideals) of one algebra; this topic will be discussed further in the appendix. Theorems 12.11–12.13 will find important applications in the present section. In the first place, we obtain with their help the following result, which constitutes an essential addition to the arithmetic of types:

THEOREM 12.14 (*GENERAL REFINEMENT THEOREM FOR ISOMORPHISM TYPES*). *If, for all $i \varepsilon I$ and $j \varepsilon J$, α_i and β_j are isomorphism types of G.C.A.'s, and if*

$$\prod_{i \varepsilon I} \alpha_i = \prod_{j \varepsilon J} \beta_j,$$

then there are isomorphism types $\gamma_{i,j}$ of G.C.A.'s such that

$$\alpha_i = \prod_{j \varepsilon J} \gamma_{i,j} \quad \text{and} \quad \beta_j = \prod_{i \varepsilon I} \gamma_{i,j} \quad \text{for all } i \varepsilon I \text{ and } j \varepsilon J.$$

Similarly for weak cardinal products.

PROOF: By 12.9

(1) $$\alpha = \prod_{i \varepsilon I} \alpha_i = \prod_{j \varepsilon J} \beta_j$$

is a type of a G.C.A. According to 12.1, let

$$\mathfrak{A} = \langle A, +, \sum \rangle$$

be a G.C.A. with

(2) $$\tau(\mathfrak{A}) = \alpha.$$

By (1), (2), and 12.11, there are ideals A_i in $\mathfrak{A}$ which satisfy conditions 12.11(i)–(iii); and also ideals B_j which satisfy the same conditions (with $I = J$ and $A_j = B_j$). By 9.6, 10.1, and 10.5, the sets $A_i \cap B_j$ are also ideals in $\mathfrak{A}$. We put

(1) $$\gamma_{i,j} = \tau(\langle A_i \cap B_j, +, \sum \rangle);$$

by 6.13 and 9.18, the types $\gamma_{i,j}$ thus defined are types of G.C.A.'s. By 12.11(i) we have

$$A = \bigcup_{i \varepsilon I} A_i = \bigcup_{j \varepsilon J} B_j;$$

$$A_i \cap A_k = I(0) \quad \text{for} \quad i, k \varepsilon I \quad \text{with} \quad i \neq k.$$

Hence we obtain by 10.9 for any given $j \varepsilon J$

(2) $$B_j = B_j \cap \bigcup_{i \varepsilon I} A_i = \bigcup_{i \varepsilon I} (A_i \cap B_j),$$

and also, in view of 10.3,

(3) $$(A_i \cap B_j) \cap (A_k \cap B_j) = I(0) \text{ for } i, k \varepsilon I \text{ with } i \neq k.$$

Since, by 9.21(ii) and 10.5, the $A_i \cap B_j$ are ideals in $\langle B_j, +, \sum \rangle$, we see from (1)–(3) that conditions (i) and (iii) of 12.11 are satisfied if we replace in them 'A' by 'B_j', and 'A_i' by '$A_i \cap B_j$'; the same applies to condition (ii) since $A_i \cap B_j$ are subsets of A_i. Hence, by 12.11,

$$\beta_j = \tau(\langle B_j, +, \textstyle\sum \rangle) = \prod_{i \varepsilon I} \gamma_{i,j}.$$

In a similar way we obtain

$$\alpha_i = \prod_{j \varepsilon J} \gamma_{i,j};$$

and the proof is complete. The proof for weak products is practically the same; instead of 12.11, we apply 12.13.

Theorem 12.14 can be extended to strong cardinal products also.

12.14 implies various corollaries concerning indecomposable types of C.A.'s and G.C.A.'s; they are analogous to certain theorems of §4 (e.g., 4.43–4.46), but apply to products of arbitrarily many factors. Thus, for instance, we can easily show that every type of a G.C.A. has, apart from order, at most one representation as a product of indecomposable types.

From theorems stated so far in this section it is seen that Postulates 1.1.I–VI are satisfied in the algebra of types of arbitrary G.C.A.'s. Our next task is to show that Postulate 1.1.VII also holds if we restrict ourselves to types of C.A.'s, and that consequently the algebra of these types is itself a C.A.

THEOREM 12.15. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a C.A., and if* T *is the set of all types of subalgebras* $\mathfrak{B} = \langle B, +, \sum \rangle$ *constituted by ideals* B *in* $\mathfrak{A}$, *then the algebra* $\langle \mathrm{T}, \times, \prod \rangle$ *is a G.C.A. and is isomorphic with* $\mathfrak{J}(\mathfrak{A})/\cong$.

PROOF: X being an ideal in $\mathfrak{A}$, and $\mathsf{X} = (X/\cong)$ the corresponding coset in $\mathfrak{J}(\mathfrak{A})/\cong$ (cf. 6.3, 8.16, and 10.1), we correlate with X the isomorphism type

(1) $$\varphi(\mathsf{X}) = \tau(\langle X, +, \textstyle\sum \rangle).$$

By 6.3 and 12.1, φ is obviously a biunique function which maps the family of all cosets X onto the set T. We show without difficulty that for any integer $n \leqq \infty$ and for any cosets $\alpha, \alpha_0, \alpha_1, \cdots, \alpha_i, \cdots$ in $\mathfrak{J}(\mathfrak{A})/\cong$ the formulas

$$(2) \qquad \alpha = \sum_{i<n} \alpha_i \quad \text{and} \quad \varphi(\alpha) = \prod_{i<n} \varphi(\alpha_i)$$

are equivalent. The proof is based essentially upon 11.36 and 12.12; we also use 6.3, 6.4, 6.13, and 10.12. By applying 6.1 (and 12.2) now, we conclude that the algebra $\langle \mathrm{T}, \times, \prod \rangle$ is isomorphic with $\mathfrak{J}(\mathfrak{A})/\cong$; hence, by 11.36, this algebra is a G.C.A.

COROLLARY 12.16. *If $\alpha_0, \alpha_1, \cdots, \alpha_i, \cdots, \beta_0, \beta_1, \cdots, \beta_i, \cdots$ are types of C.A.'s and $\alpha_n = \beta_n \times \alpha_{n+1}$ for every $n < \infty$, then there is a type γ of a C.A. such that*

$$\alpha_n = \gamma \times \prod_{i<\infty} \beta_{n+i} \quad \textit{for every} \quad n < \infty.$$

PROOF: By 12.1, 12.2, and 12.9 there is a C.A. $\mathfrak{A}$ with

$$\tau(\mathfrak{A}) = \prod_{i<\infty} \alpha_i \times \prod_{i<\infty} \beta_i;$$

and, by 12.5 and 12.11, α_i and β_i are types of certain subalgebras of $\mathfrak{A}$ constituted by ideals. Hence, by 12.15, we can apply 5.1.V (with '+' replaced by '×') to α_i and β_i; and, in view of 12.9, we obtain the conclusion.

THEOREM 12.17 (FUNDAMENTAL THEOREM OF THE ALGEBRA OF ISOMORPHISM TYPES). *T being the class of all types of C.A.'s, the algebra $\mathfrak{T} = \langle \mathrm{T}, \times, \prod \rangle$ is itself a C.A.*

PROOF: by 1.1, 12.2, 12.4, 12.5, 12.7(i), 12.9(i), 12.10, 12.14, and 12.16.

It will be seen from 12.18 that Theorems 12.15–12.17 cannot be extended to types of G.C.A.'s or even of finitely closed G.C.A.'s. It may be noticed that the only reason why the proof of 12.15 does not apply to G.C.A.'s is that condition (ii) in 12.12 is automatically satisfied and hence can be omitted in case $\mathfrak{A}$ is a C.A.—while the same condition is essential if $\mathfrak{A}$ is an arbitrary G.C.A.

THEOREM 12.18.[6] *Let T_1, or T_2, be the class of all types of finitely closed G.C.A.'s, or of arbitrary G.C.A.'s. Then*

[6] The first part of Theorem 12.18 is due to B. Jónsson; the proof outlined here uses essentially Jónsson's idea.

(i) *the algebras* $\mathfrak{T}_1 = \langle \mathrm{T}_1, \times, \prod \rangle$ *and* $\mathfrak{T}_2 = \langle \mathrm{T}_2, \times, \prod \rangle$ *are not G.C.A.'s;*

(ii) *more generally, no operation* $\prod'$ *on infinite sequences of types can be defined for which the algebra* $\langle \mathrm{T}_1, \times, \prod' \rangle$, *or* $\langle \mathrm{T}_2, \times, \prod' \rangle$, *is a G.C.A.* (*the symbol* '$\times$' *being used in its original meaning*).

Proof: Assume, to the contrary, that there is an operation $\prod'$ for which the algebra

$$\mathfrak{T}' = \langle \mathrm{T}_1, \times, \textstyle\prod' \rangle \quad (\text{or} \quad \mathfrak{T}' = \langle \mathrm{T}_2, \times, \textstyle\prod' \rangle)$$

is a G.C.A.

We construct a C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$ in which the set A contains just two distinct elements, say, z and u; for this purpose, we put:

$$a + b = z \text{ if } a = b = z, \text{ and } a + b = u \text{ otherwise};$$

$$\sum_{i<\infty} a_i = z \text{ if } a_i = z \text{ for every } i < \infty, \text{ and } \sum_{i<\infty} a_i = u \text{ otherwise};$$

where $a, b, a_0, a_1, \cdots, a_i, \cdots$ are arbitrary elements of A. Let α be the type of $\mathfrak{A}$. As is easily seen from 6.11, 12.1, 12.2, 12.6, and 12.8,

(1) $$\alpha \text{ is indecomposable.}$$

We now put:

(2) $\beta = \prod_{i<\infty} \alpha$ (i.e., $\beta = \prod_{i<\infty} \alpha_i$ where $\alpha_i = \alpha$ for $i = 0, 1, 2, \cdots$)

and

(3) $$\beta' = \prod_{i<\infty}{}_w\, \alpha.$$

By 12.9, β is the type of a C.A., and β' is the type of a finitely closed G.C.A., but not of a C.A.; hence

(4) $$\beta \neq \beta'.$$

At any rate, however, α, β, and β' are in T_1 (and in T_2). From (2) and (3) we obtain by 12.5

(5) $$\beta = \alpha \times \beta \quad \text{and} \quad \beta' = \alpha \times \beta'.$$

Hence, by applying 5.1.V (with $a_n = \beta$, or $= \beta'$, and $b_n = \alpha$) to the algebra $\mathfrak{T}'$, we conclude that the type

(6) $$\gamma = \prod_{i<\infty}{}' \alpha$$

exists and belongs to T_1, and that

(7) $\beta = \gamma \times \delta$ and $\beta' = \gamma \times \delta'$ for some $\delta, \delta' \varepsilon T_1$.

Moreover, by applying 5.1.I to $\mathfrak{T}'$, we obtain from (6)

(8) $$\gamma = \alpha \times \gamma.$$

By (2) and (7) we have

$$\gamma \times \delta = \prod_{i<\infty} \alpha = \beta.$$

Hence, by 12.2, 12.5, 12.7, 12.8, 12.14, and (1), we easily conclude that

(9) $\gamma = \prod_{i<n} \alpha$ for some $n < \infty$, or else $\gamma = \beta$.

Similarly, (1), (3), and (7) imply

(10) $\gamma = \prod_{i<n}^{w} \alpha$ for some $n < \infty$, or else $\gamma = \beta'$.

(4), (9), and (10) give in view of 12.3(ii)

$$\gamma = \prod_{i<n} \alpha \quad \text{for some} \quad n < \infty.$$

Hence and from (8) we infer by 4.3 that α is idem-multiple, i.e., that

$$\alpha = \alpha \times \alpha.$$

This, however, clearly contradicts (1) (cf. 4.39 or 12.8). Thus, we must reject our original assumption; and the proof is complete.

Although the algebra $\mathfrak{T}_1$ from 12.18 is not a G.C.A., it has nevertheless some interesting arithmetical properties; for, as we shall show in 12.21, it proves to be an R.A.

THEOREM 12.19. *If* $\mathfrak{A} = \langle A, +, \sum \rangle$ *is a finitely closed G.C.A., and if* T *is the set of all types of subalgebras* $\mathfrak{B} = \langle B, +, \sum \rangle$ *constituted by ideals* B *in* $\mathfrak{A}$, *then the algebra* $\langle \mathrm{T}, \times, \prod \rangle$ *is an R.A.*

PROOF: We proceed as in the proof of 12.15; i.e., we define the function φ by means of formula (1) of that proof, and we show that φ maps in a one-to-one way the family of all cosets of the algebra $\mathfrak{J}(\mathfrak{A})/\cong$ onto the set T. However, formulas (2) now prove to be equivalent only in case $n < \infty$. (This depends on the fact that condition 12.12(ii) is now automatically satisfied only in the case

of a finite sequence of ideals.) In other words, φ maps $\langle \mathrm{T}, \times, \prod \rangle$ isomorphically onto $\mathfrak{J}(\mathfrak{A})/\cong$ only in the domain of binary operations; i.e., it satisfies conditions 6.1(i)–(iii) without necessarily satisfying 6.1(iv). This is, however, sufficient for our purposes. For $\mathfrak{J}(\mathfrak{A})/\cong$ is an R.A. by 11.36; and since in the postulates characterizing R.A.'s only the binary operation is involved (cf. 11.26), we conclude that $\langle \mathrm{T}, \times, \prod \rangle$ is also an R.A.

Corollary 12.20. *If α_1, α_2, β, and γ are types of finitely closed G.C.A.'s, and if $\alpha_1 \times \alpha_2 \times \gamma = \beta \times \gamma$, then there are types β_1, β_2, γ_1, and γ_2 such that*

$$\beta = \beta_1 \times \beta_2, \quad \gamma = \gamma_1 \times \gamma_2, \quad \alpha_1 \times \gamma_1 = \beta_1 \times \gamma_1, \quad \textit{and}$$

$$\alpha_2 \times \gamma_2 = \beta_2 \times \gamma_2 .$$

Proof: analogous to that of 12.16, with 12.15 and 5.1.V replaced by 12.19 and 11.26.V.

Theorem 12.21. *T_1 being the class of all types of finitely closed G.C.A.'s, the algebra $\mathfrak{T}_1 = \langle \mathrm{T}_1, \times, \prod \rangle$ is a finitely closed R.A.*

Proof: by 12.2, 12.4, 12.5, 12.7(i), 12.9(i), 12.10, 12.14, and 12.20.

Thus, all the arithmetical theorems listed in 11.28 hold in the algebra $\mathfrak{T}_1$; they do not have to be provided with existential assumptions (required in arbitrary R.A.'s) since the algebra $\mathfrak{T}_1$ is finitely closed. Among these theorems, 1.31 and 2.28 are perhaps the most interesting. By 1.31, the factor relation—which is obviously reflexive and transitive in the class of all isomorphism types—proves to be antisymmetric when applied to types of the class T_1, and hence to establish in T_1 a partial order; by 2.28, this relation has in addition the interpolation property. Notice that an alternative formulation of 1.31 (1.47 with $n = 2$) gives:

If α, β, and γ are types of finitely closed G.C.A.'s and $\alpha \times \beta \times \gamma = \gamma$, then $\alpha \times \gamma = \beta \times \gamma = \gamma$.

As regards the algebra $\mathfrak{T}_2$ of 12.18, i.e., the algebra of types of arbitrary G.C.A.'s, no general arithmetical results are known which hold in this algebra and are not more or less trivial consequences of elementary theorems stated in the earlier part of this section (with 12.14 included). We are lacking, however, in counter-examples which would definitely show that various theorems applying to the algebra $\mathfrak{T}_1$ by virtue of 12.21 actually fail in the algebra $\mathfrak{T}_2$;

it would be interesting to know, for instance, whether or not the factor relation establishes a partial order in T_2. Nor do we know whether various special theorems of Part I which are not listed in 11.28 hold in the algebras $\mathfrak{T}_1$ and $\mathfrak{T}_2$—although, by 12.17, they certainly hold in the algebra $\mathfrak{T}$. This applies, e.g., to Theorem 2.34; we do not know whether for all types α and β of G.C.A.'s, or of finitely closed G.C.A.'s, $\alpha \times \alpha = \beta \times \beta$ implies $\alpha = \beta$. Finally, there are problems of the same nature which remain open even with regard to the algebra $\mathfrak{T}$. It is not known, for instance, whether any two types α and β of C.A.'s have a LEAST COMMON MULTIPLE, i.e., a least upper bound in the algebra $\mathfrak{T}$; we shall see later that two such types do not always have a GREATEST COMMON FACTOR, i.e., a greatest lower bound in the algebra $\mathfrak{T}$.

It can be shown that the algebra $\mathfrak{T}$ of 12.17 is not only a subalgebra but also a homomorphic image of the algebras $\mathfrak{T}_1$ and $\mathfrak{T}_2$ of 12.18; and this remains true even in case we regard the operation $\prod$ in these algebras as applying to sets of elements with an arbitrary power. In fact, α being the type of a G.C.A. $\mathfrak{A}$, let $\bar{\alpha}$ be the type of all C.A.'s $\bar{\mathfrak{A}}$ which are closures of $\mathfrak{A}$; and let R be the relation which holds between two types α and β of G.C.A.'s if, and only if, $\bar{\alpha} = \bar{\beta}$. By 7.13, $\bar{\alpha}$ is uniquely determined by α. By 17.17 we have, for any types α_i of G.C.A.'s,

$$\overline{\prod_{i \in I} \alpha_i} = \prod_{i \in I} \overline{\alpha_i}.$$

Hence we conclude that R is an infinitely additive (or rather MULTIPLICATIVE) equivalence relation both in $\mathfrak{T}_1$ and $\mathfrak{T}_2$, and that

$$\mathfrak{T} \cong \mathfrak{T}_1/R \cong \mathfrak{T}_2/R.$$

R can also be shown to be finitely refining in $\mathfrak{T}_1$ (but not in $\mathfrak{T}_2$). These results, however, do not provide us with any interesting information about the algebras $\mathfrak{T}_1$ and $\mathfrak{T}_2$.

In view of 12.3(ii), Theorems 12.15–12.21 apply automatically to strong cardinal products. It is also obvious that Theorems 12.18–12.21 apply to weak products as well; although 12.15–12.17 fail in this case (as is seen from the last part of 12.9). It may be noticed that if we replace in the algebras $\mathfrak{T}_1$ and $\mathfrak{T}_2$ plain products by weak ones, then $\mathfrak{T}_1$ proves to be a homomorphic image of $\mathfrak{T}_2$; to show this, we proceed as above in the case of $\mathfrak{T}$—using, however,

the notion of a finite closure instead of that of a closure (cf. 7.18 and the subsequent remarks).

The last three theorems of this section are of a more special nature since they apply to isomorphism types of rather restricted classes of C.A.'s.

THEOREM 12.22. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a C.A. in which every element* $a \in A$ *can be represented in the form*

$$a = \sum_{i<n} a_i \quad \textit{with} \quad n \leqq \infty$$

where all elements a_i *with* $i < n$ *are indecomposable in the disjunctive algebra* $\dot{\mathfrak{A}}$. *Then* $\tau(\mathfrak{A})$ *can be represented in the form*

$$\tau(\mathfrak{A}) = \prod_{i \in I} \alpha_i$$

where all types α_i *with* $i \in I$ *are indecomposable; and this representation is unique apart from order.*

PROOF: By 10.16 and 12.11 we obtain a representation of $\tau(\mathfrak{A})$:

$$\tau(\mathfrak{A}) = \prod_{i \in I} \alpha_i \quad \text{with} \quad \alpha_i = \tau(\langle A_i, +, \sum \rangle) \quad \text{for every} \quad i \in I,$$

where the A_i are ideals in $\mathfrak{A}$ which are indecomposable in the disjunctive ideal algebra $\dot{\mathfrak{J}}(\mathfrak{A})$. Hence we easily conclude that all the types α_i are indecomposable in the sense of 12.8; we apply here in the first place 12.12, and also 8.16, 9.12, 10.1, 10.3, and 12.6. The unicity of representation follows easily from 10.16 and 12.11, or else it can be derived from 12.5, 12.7, 12.8, and 11.14 (cf. the proof of 4.45).

As was pointed out in connection with 10.16, the class of C.A.'s involved in 12.22 contains as a subclass the class of those C.A.'s $\mathfrak{A}$ in which every element can be represented as a sum of elements that are indecomposable in $\mathfrak{A}$ itself (and not only in $\dot{\mathfrak{A}}$). We shall see in 14.10 that, with regard to algebras of the latter class, the conclusion of 12.22 can be strengthened considerably; the type of every such algebra is a cardinal power (i.e., a product with identical factors) of a well-determined indecomposable type α, in fact, of the type of the algebra of non-negative integers.

THEOREM 12.23. *Let* $\mathfrak{A} = \langle A, +, \sum \rangle$ *be a C.A. in which there is no infinite sequence of elements* $b_0, b_1, \cdots, b_i, \cdots$ *with* $b_i \neq 0$

and $b_i \cap b_j = 0$ *for* $i < j < \infty$. *Then* $\tau(\mathfrak{A})$ *can be represented in the form*

$$\tau(\mathfrak{A}) = \prod_{i<n} \alpha_i \quad \textit{with} \quad n < \infty$$

where all types α_i *with* $i < n$ *are indecomposable; and this representation is unique, apart from order.*

PROOF: entirely analogous to that of 12.22, with 10.16 replaced by 10.17.

COROLLARY 12.24. *The conclusion of* 12.23 *applies to all at most denumerable C.A.'s.*

PROOF: by 10.18 and 12.23.

We shall see in the appendix that 12.23 is but a particular case of a theorem of general algebra, and that the validity of its conclusion does not depend on any specific properties of C.A.'s. In particular, 12.23 (but not 12.24) can be extended to arbitrary G.C.A.'s, although this is not directly seen from the proof of the theorem; cf. the remark which follows 12.17. Some other theorems of general algebra which concern the decomposition of types into indecomposable factors and can be applied to C.A.'s and G.C.A.'s will also be found in the appendix.

In connection with 12.22–12.24 the problem arises whether every type of a C.A. can be represented as a product of indecomposable factors. It can easily be shown that this is not the case. We can even obtain a much stronger result: there are types α of C.A.'s which are different from 1 and have no indecomposable factors at all. In fact, we can construct a C.A. $\mathfrak{A} = \langle A, +, \sum \rangle$ with the following properties: there are elements $a \neq 0$ and $b \neq 0$ in A such that $a \cap b = 0$ and $c \leqq a + b$ for every $c \in A$; and any two elements $a \neq 0$ and $b \neq 0$ in A are homogeneous in the sense of 11.30. (It is well known that there are complete Boolean algebras with these properties, e.g., the Boolean algebra of all regular open sets of a Euclidean space under set-theoretical addition[7]; and, as we shall see in §15, every complete Boolean algebra is a C.A.) The type α of such a C.A. $\mathfrak{A}$ clearly satisfies the following conditions:

$$\alpha = \alpha \times \alpha \neq 1; \quad \text{and} \quad \alpha = \zeta \times \eta \quad \text{implies that} \quad \zeta = 1$$
$$\text{or} \quad \zeta = \alpha.$$

[7] This follows from Theorem 7.23 and 7.26 in Tarski [7], p. 178 f.

Hence it follows at once that α has no indecomposable factor. It should be pointed out, however, that α is itself indecomposable in a weaker sense mentioned in connection with 4.38; and the problem remains open whether every type $\alpha \neq 1$ of a C.A. has a factor which is indecomposable in this weaker sense.

In conclusion, we want to discuss briefly certain variants of the notion of a cardinal product, which are different from those defined in 6.11 and 12.2. We have to use here the familiar notion of a cardinal number (which will be discussed systematically in §17—in its relation to the general theory of C.A.'s). Let ν be any cardinal different from 0; and let $\mathfrak{A}_i$ be algebras with zero elements 0_i correlated with elements i of a set I. By the CARDINAL PRODUCT of the algebras $\mathfrak{A}_i$ WITH THE RANK ν we understand the subalgebra of the strong cardinal product of $\mathfrak{A}_i$ constituted by those functions f for which the set of all elements $i \varepsilon I$ with $f(i) \neq 0_i$ has a power smaller than ν (cf. 6.11). We extend this notion to isomorphism types by changing Definition 12.2 correspondingly.

The cardinal product with the rank $\nu = \aleph_0$ is simply the weak cardinal product, and that with the rank $\nu = \aleph_1$ coincides with the plain cardinal product; here $\aleph_0$ is, as always, the smallest infinite cardinal, and $\aleph_1$ is the smallest cardinal $> \aleph_0$. Cardinal products with a rank $\nu > \aleph_1$ have practically the same properties with regard to G.C.A.'s as the strong product; we do not want to go into certain details here connected with the extension of 6.12 and 12.9 to such products. They are useful, however, as a source of various examples. Consider, for instance, the cardinal $\nu = \aleph_\Omega$, i.e., the smallest cardinal ν such that the set of cardinals $< \nu$ is non-denumerable. Let α be the type of a C.A. with just two different elements (cf. the proof of 12.18); let β be the strong cardinal product of ν factors each of which equals α, and let β' be the cardinal product of the same factors with the rank ν. By arguing as in the proof of 12.18, we can show that β and β' constitute an example of two types of C.A.'s which have no greatest common factor.[8] It may be noticed that the types β and β' constructed in the proof of 12.18 have also the same property; in this case, however, β' is not the type of a C.A.

As regards cardinal products with a finite rank ν, the case $\nu = 1$ is trivial; the case when $2 < \nu < \aleph_0$ seems to be unnatural and can hardly lead to any interesting results. The remaining case $\nu = 2$

[8] This example was found by B. Jónsson.

deserves some attention only if we happen to be interested in algebras which, like G.C.A.'s, are not assumed to satisfy the closure postulates; for, as is easily seen, the cardinal product of algebras $\mathfrak{A}_i$ with the rank 2 is never finitely closed unless there is at most one i such that $\mathfrak{A}_i$ has two or more distinct elements. However, the cardinal product with the rank 2 of arbitrary G.C.A.'s is always a G.C.A. This notion is very closely related to that of cardinal addition which will be discussed in §18, and has the same fundamental properties; the class of all isomorphism types—not only of types of G.C.A.'s—proves to be a C.A. under this sort of cardinal multiplication, and every type can be uniquely represented as a product with the rank 2 of indecomposable factors. (Of course, 'indecomposable' means something else in this context than in 12.8, for the product with the rank 2 does not coincide with the ordinary product even in the case of two factors.) The proof of these results will not be outlined here; the proofs of Theorems 18.6 and 18.7 below give a very adequate idea of the method to be applied.

PART III

CONNECTIONS WITH OTHER ALGEBRAIC SYSTEMS

§ 13. SEMIGROUPS

In this part of our work we want to discuss relations between C.A.'s and G.C.A.'s on the one hand, and certain algebraic systems well known from the literature on the other. In most cases it proves convenient here to regard C.A.'s and G.C.A.'s as algebras with a single operation, in fact, with binary addition.

DEFINITION 13.1. *An algebra* $\mathfrak{A} = \langle A, + \rangle$ *is called a C.A., or a G.C.A., if there is an operation* $\sum$ *on infinite sequences of elements of A such that* $\langle A, +, \sum \rangle$ *is a C.A. in the sense of* 1.1, *or a G.C.A. in the sense of* 5.1. (*Under the same conditions we say that A is a C.A., or a G.C.A.,* UNDER THE OPERATION $+$.)

In addition to C.A.'s and G.C.A.'s, we shall be interested in various other algebras $\mathfrak{A}$ constituted by a set A and a binary operation $+$. The following definition applies to arbitrary algebras of this kind:

DEFINITION 13.2. *We extend to algebras* $\mathfrak{A} = \langle A, + \rangle$ *the definitions of the element* 0, *the relation* $\leqq$, *the greatest lower bound, and the least upper bound* (1.2, 1.5, 3.1, *and* 3.2). *We define the sum of a sequence of elements* $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$ *by putting*

$$\sum_{i<0} a_i = 0 \text{ (in case the element 0 is in } A\text{)},$$

$$\sum_{i<n+1} a_i = \sum_{i<n} a_i + a_n \quad \text{for} \quad n < \infty \quad \text{(in case } \sum_{i<n} a_i \text{ and } \sum_{i<n} a_i + a_n \text{ are in } A\text{)},$$

$$\sum_{i<\infty} a_i = \bigcup_{n<\infty} \sum_{i<n} a_i \text{ (in case all sums } \sum_{i<n} a_i \text{ with } n < \infty \text{ are in } A \text{ and their least upper bound exists).}$$

Furthermore, we extend to arbitrary algebras $\mathfrak{A} = \langle A, + \rangle$ *the definitions of the remaining notions introduced in* §§1–12, *by omitting in these definitions—except* 1.4, 6.4, 6.7, 7.1, *and* 9.1—*all references to infinite operations.*

Thus, for instance, by applying 8.7 and 8.16, we can construct from an algebra $\mathfrak{A} = \langle A, + \rangle$ two new algebras, $\breve{\mathfrak{A}} = \langle A, \cup \rangle$ and $\dot{\mathfrak{A}} = \langle A, \dot{+} \rangle$. On the other hand, when speaking of ideals in an algebra $\mathfrak{A} = \langle A, + \rangle$, we shall use the term 'ideal' in exactly the same sense in which it was originally defined in 9.1.

It may be noticed that we shall be concerned merely with algebras $\mathfrak{A} = \langle A, + \rangle$ in which the operation $+$ is commutative; this explains why we have not introduced in 13.2 two different relations $\leqq$. Definitions 3.1 and 3.2, as well as the definition of an infinite sum (stated in 13.2), will be applied almost exclusively to those algebras $\mathfrak{A}$ in which the set A is partially ordered by the relation $\leqq$.

COROLLARY 13.3. *If* $\mathfrak{A} = \langle A, + \rangle$ *is a C.A. or a G.C.A., then*

(i) *the operation* $\sum$ *on finite sequences defined in* 13.2 *coincides with that defined in* 1.3;

(ii) *the operation* $\sum$ *on infinite sequences defined in* 13.2 *is the only operation which satisfies the conditions of* 13.1.

PROOF: (i) by 1.8 and 1.17; (ii) by (i) and 3.19.

The simplest examples of G.C.A.'s can be found among SEMIGROUPS and LATTICES. In this section we shall concern ourselves with semigroups; the discussion of lattices will be found in §15.

DEFINITION 13.4. *Let* $\mathfrak{A} = \langle A, + \rangle$.

(i) $\mathfrak{A}$ *is called a* GROUPOID *(or, strictly speaking, a* COMMUTATIVE GROUPOID*) if it satisfies the closure postulate* 1.1.I, *the postulate of the zero element* 1.1.V, *and the commutative and associative laws* 1.13 *and* 1.14.

(ii) $\mathfrak{A}$ *is called a* SEMIGROUP *(or, strictly speaking, a* COMMUTATIVE SEMIGROUP*) if it is a groupoid and satisfies the cancellation law for sums:*

If $a, b, c \in A$ *and* $a + c = b + c$, *then* $a = b$.

COROLLARY 13.5. *In a semigroup* $\mathfrak{A} = \langle A, + \rangle$ *all elements of* A *are finite.*

PROOF: by 13.2 (4.10) and 13.4.

In our further discussion we shall often apply 13.1–13.4 without referring to them explicitly; 13.1 and 13.2 will, of course, be implicitly involved in all those cases in which we shall use definitions and theorems of §§1–12. Various elementary consequences of 13.4 are assumed to be known. It is easily seen, for instance, that most of the theorems stated in §1, when restricted to finite sums and multiples, apply to arbitrary groupoids and semigroups.

Every finitely closed G.C.A. is obviously a groupoid. Moreover, we have

THEOREM 13.6. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be a G.C.A. For* $\mathfrak{A}$ *to be a semigroup it is necessary and sufficient that* $\mathfrak{A}$ *be finitely closed and that every element in* A *be finite.*

PROOF: by 4.19, 5.26, and 13.5.

COROLLARY 13.7. *No C.A.* $\mathfrak{A} = \langle A, + \rangle$ *is a semigroup unless* A *contains only one element.*

PROOF: by 4.12 and 13.6.

THEOREM 13.8. *If* $\mathfrak{A} = \langle A, + \rangle$ *is a C.A. or a finitely closed G.C.A, and if* B *is the set of all finite elements of* A, *then* $\mathfrak{B} = \langle B, + \rangle$ *is a G.C.A. and a semigroup.*

PROOF: By 4.10, 4.13, 4.14, 4.16, 6.13, 9.1, and 9.18, $\mathfrak{B}$ is a finitely closed G.C.A., and every element of $\mathfrak{B}$ is finite; hence, by 13.6, $\mathfrak{A}$ is also a semigroup.

COROLLARY 13.9. *For a C.A.* $\mathfrak{A} = \langle A, + \rangle$ *to be a closure of a semigroup, it is necessary and sufficient that every element* $a \in A$ *can be represented in the form*

$$a = \sum_{i<\infty} a_i$$

where all elements a_i *with* $i < \infty$ *are finite.*

PROOF: by 7.1, 7.6, 13.6, and 13.8.

We know from 13.6 under what conditions a given G.C.A. is a semigroup. More interesting is the question under what conditions a given semigroup is a G.C.A. To answer this question, we shall first distinguish certain special kinds of semigroups.

DEFINITION 13.10. *An algebra* $\mathfrak{A} = \langle A, + \rangle$ *is called* PARTIALLY ORDERED, *or* SIMPLY ORDERED, *or* WELL ORDERED, *if the set* A *is partially ordered, or simply ordered, or well ordered, by the relation* $\leqq$.

This definition, like 13.1 or 13.4, will often be used without explicit reference.

COROLLARY 13.11. *Let* $\mathfrak{A} = \langle A, + \rangle$.

(i) *In case* $\mathfrak{A}$ *is a groupoid, it is partially ordered if, and only if,* $a + b + c = c$ *implies* $a + c = b + c = c$ *for all* $a, b, c \in A$.

(ii) *In case* $\mathfrak{A}$ *is a semigroup, it is partially ordered if, and only if,* $a + b = 0$ *implies* $a = b = 0$ *for all* $a, b \in A$.

PROOF: by 13.10, without any difficulty.

In the next few theorems we state some elementary, though less familiar, properties of partially ordered semigroups.

THEOREM 13.12. *Let $\mathfrak{A} = \langle A, + \rangle$ be a partially ordered groupoid. For any elements a_i, $b_i \varepsilon A$ correlated with elements i of a non-empty set I, and for any elements $a, b, c \varepsilon A$, we have:*

(i) *if $a_i + b_i = c$ for every $i \varepsilon I$, and $\bigcup_{i\varepsilon I} b_i$ exists, then $\bigcap_{i\varepsilon I} a_i$ also exists and*

$$\bigcap_{i\varepsilon I} a_i + \bigcup_{i\varepsilon I} b_i = c;$$

(ii) *if $a \cup b$ exists, then $a \cap b$ also exists, and $(a \cap b) + (a \cup b) = a + b$.*

PROOF: (i) can be derived in the same way as the corresponding part of 4.21 (since 4.20 obviously holds for arbitrary elements of a semigroup). (ii) is a particular case of (i); cf. the proof of 4.22.

COROLLARY 13.13. *If $\mathfrak{A} = \langle A, + \rangle$ is a partially ordered semigroup, $a, b, c \varepsilon A$, $a \cap c = b \cap c$, and $a \cup c = b \cup c$, then $a = b$.*

PROOF: by 13.12(ii).

As is known, the distributive laws 3.30 and 3.32 with $n < \infty$ can be derived from 13.13 if provided with suitable existential assumptions; it seems that these assumptions must be stronger than those actually stated in 3.30 and 3.32.[1] Two other distributive laws, 3.25 and 3.26, also hold for semigroups, and can even be generalized to sets of an arbitrary power. This will be shown in the following

THEOREM 13.14. *Let $\mathfrak{A} = \langle A, + \rangle$ be a partially ordered semigroup; let a be in A, and let b_i be elements of A correlated with elements i of an arbitrary non-empty set I. We then have:*

(i) *if either $\bigcup_{i\varepsilon I} b_i$ or $\bigcup_{i\varepsilon I} (a + b_i)$ exists, then both these bounds exist, and*

$$a + \bigcup_{i\varepsilon I} b_i = \bigcup_{i\varepsilon I} (a + b_i);$$

(ii) *if $\bigcap_{i\varepsilon I} (a + b_i)$ exists, then $\bigcap_{i\varepsilon I} b_i$ also exists, and*

$$a + \bigcap_{i\varepsilon I} b_i = \bigcap_{i\varepsilon I} (a + b_i).$$

[1] By a theorem in Bergmann [1], p. 273, the distributive laws 3.30 and 3.32 with $n < \infty$ can be derived from 13.13 in case the semigroup $\mathfrak{A}$ is lattice-ordered in the sense of 13.22. An analysis of Bergmann's proof shows that his result can be extended to arbitrary partially ordered systems by providing the distributive laws with sufficiently strong existential assumptions.

PROOF: (i) Assume, e.g., $\bigcup_{i\varepsilon I}(a + b_i)$ to exist. By 3.2, we have, for every $i \varepsilon I$ and for some $c \varepsilon A$,

$$a + b_i \leqq \bigcup_{i\varepsilon I} (a + b_i) = a + c, \tag{1}$$

and hence

$$b_i \leqq c. \tag{2}$$

If, for every $i \varepsilon I$,

$$b_i \leqq x,$$

we also have

$$a + b_i \leqq a + x;$$

therefore, by (1),

$$a + c \leqq a + x \quad \text{and} \quad c \leqq x.$$

Hence, and in view of (2),

$$c = \bigcup_{i\varepsilon I} b_i. \tag{3}$$

(1) and (3) imply the conclusion. The proof of (i) under the alternative assumption is essentially the same; and the derivation of (ii) is quite analogous.

It is easily seen that the converses of 13.12(ii) and 13.14(ii) may fail in partially ordered semigroups. In fact, let I be the set of all negative and positive integers, with -1 (and 0) excluded. I is clearly a partially ordered semigroup under ordinary multiplication. We have in this semigroup $2 \cap 3 = 1$; however, neither $2 \cup 3$ nor $(a\cdot 2) \cap (a\cdot 3)$ for any $a \varepsilon I$, $a \neq 1$, exists. Notice in this connection:

THEOREM 13.15. *Let $\mathfrak{A} = \langle A, + \rangle$ be a partially ordered semigroup, and let a and b be any elements of A. For $a \cup b$ to exist it is necessary and sufficient that $(a + x) \cap (b + x)$ exist for every $x \varepsilon A$.*

PROOF: The existence of $a \cup b$ implies that of $(a + x) \cap (b + x)$ by 13.12(ii) and 13.14(i). Assume now $(a + x) \cap (b + x)$ to exist for every $x \varepsilon A$. Thus, in particular, $a \cap b$ exists, and we can put (by 3.1)

$$(a \cap b) + c = a + b;$$

hence we obtain

$$a \leqq c \quad \text{and} \quad b \leqq c.$$

Now if

$$a \leqq x \quad \text{and} \quad b \leqq x,$$

then, by our assumption and 13.14(ii),

$$(a \cap b) + c = a + b \leqq (a + x) \cap (b + x) = (a \cap b) + x,$$

and consequently

$$c \leqq x.$$

Thus, by 3.2,

$$c = a \cup b;$$

and the proof is complete.

THEOREM 13.16. *Let $\mathfrak{A} = \langle A, + \rangle$ be a partially ordered semigroup, and let $a_0, a_1, \cdots, a_i, \cdots, b_0, b_1, \cdots, b_i, \cdots$ be any elements in A.*

(i) *If $\sum_{i<\infty} a_i$ or $\sum_{i<\infty} a_{i+1}$ is in A, then both these sums are in A, and we have*

$$\sum_{i<\infty} a_i = a_0 + \sum_{i<\infty} a_{i+1}.$$

(ii) *If $\sum_{i<\infty} a_i, \sum_{i<\infty} b_i \,\varepsilon\, A$, then*

$$\sum_{i<\infty} (a_i + b_i) = \sum_{i<\infty} a_i + \sum_{i<\infty} b_i.$$

PROOF: In view of 13.2, (i) is but a particular case of 13.14(i); we take for I the set of all finite integers, and put

$$a = a_0, \quad \text{and} \quad b_n = \sum_{i<n} a_{i+1} \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

As regards (ii), it is easily seen that

$$\sum_{i<n} (a_i + b_i) \leqq \sum_{i<\infty} a_i + \sum_{i<\infty} b_i \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

Now if

$$\sum_{i<n} (a_i + b_i) \leqq x \quad \text{for every} \quad n < \infty,$$

then we clearly have

$$\sum_{i<n} a_i + \sum_{i<p} b_i \leqq x \quad \text{for} \quad n, p = 0, 1, 2, \cdots;$$

and, by applying 13.14(i) twice, we obtain

$$\sum_{i<\infty} a_i + \sum_{i<\infty} b_i \leqq x.$$

The conclusion of (ii) follows by 3.2.

THEOREM 13.17. *Let $\mathfrak{A} = \langle A, + \rangle$ be a partially ordered semigroup; let n be $\leqq \infty$, and let $a_0, a_1, \cdots, a_i, \cdots$ be elements in A such that $a_i \cup a_j = a_i + a_j$ for all i and j with $i < j < n$. We then have*

$$\bigcup_{i<n} a_i = \sum_{i<n} a_i$$

in case $n < \infty$; and the same holds in case $n = \infty$, provided $\sum_{i<\infty} a^i$ or $\bigcup_{i<\infty} a_i$ is in A.

PROOF: The conclusion obviously holds for $n = 0$. Assume it to hold for a given $n = p < \infty$. We then obtain with the help of 3.2, 13.14(i), and the hypothesis

$$\sum_{i<p+1} a_i = \bigcup_{i<p} a_i + a_p = \bigcup_{i<p} (a_i + a_p) = \bigcup_{i<p} (a_i \cup a_p) = \bigcup_{i<p+1} a_i .$$

Thus, the conclusion proves to hold for $n = p + 1$. By 13.2, the result can be extended at once to the case $n = \infty$.

Notice that the condition

$$a_i \cup a_j = a_i + a_j ,$$

which occurs in the hypothesis of 13.17, implies by 13.12 in every partially ordered semigroup

$$a_i \cap a_j = 0.$$

The implication in the opposite direction fails in general; it holds in those special semigroups and groupoids which we are going to discuss now.

DEFINITION 13.18. *An algebra $\mathfrak{A} = \langle A, + \rangle$ is said to have the* REFINEMENT PROPERTY *if it satisfies Postulate* 11.26.IV.

Groupoids with the refinement property are closely related to what we have called refinement algebras (cf. 11.26). In fact, we have

COROLLARY 13.19. *Finitely closed R.A.'s $\mathfrak{A} = \langle A, + \rangle$ coincide with partially ordered groupoids which have the refinement property and satisfy* 11.26.V.

PROOF: by 5.26, 11.26, 13.10, and 13.18; it follows from 11.28 that every R.A. is partially ordered.

COROLLARY 13.20. *Every partially ordered semigroup $\mathfrak{A}$ with the refinement property is a finitely closed R.A.*

PROOF: by 13.19. Postulate 11.26.V is obviously satisfied in every semigroup $\mathfrak{A}$; to obtain elements which satisfy the conclusion of 11.26.V, we put

$$b_1 = a_1, \quad b_2 = a_2, \quad c_1 = c, \quad \text{and} \quad c_2 = 0.$$

By 13.20, all theorems listed in 11.28 hold in every partially ordered semigroup with the refinement property.[2] It is easily seen that most of these theorems apply to a wider class of algebras—in fact, to arbitrary partially ordered groupoids with the refinement property, even to those which do not satisfy 11.26.V. There are, however, a few important exceptions: 2.18, 2.27, 2.28, 2.30, 3.16, and 3.17.

THEOREM 13.21. *For a semigroup $\mathfrak{A} = \langle A, + \rangle$ to have the refinement property, it is necessary and sufficient that the interpolation theorem* 2.28 *with $n = p = 2$ hold for arbitrary elements of A.*

PROOF: By applying 13.18 and by analyzing the proof of 2.28 for $n = p = 2$ we easily see that the condition of our theorem is necessary. Now assume this condition to be satisfied; and let a_1, a_2, b_1, and b_2 be any elements in A for which

$$(1) \qquad a_1 + a_2 = b_1 + b_2 .$$

We have thus

$$a_2 \leqq a_1 + a_2, \quad b_1 \leqq a_1 + a_2, \quad a_2 \leqq b_1 + a_2, \quad \text{and} \quad b_1 \leqq b_1 + a_2 .$$

Hence, by 2.28, there is an element c_1 such that

$$(2) \qquad b_1 \leqq c_1 + a_2 \leqq a_1 + a_2 \quad \text{and} \quad c_1 + a_2 \leqq b_1 + a_2 .$$

Consequently,

$$c_1 \leqq a_1 \quad \text{and} \quad c_1 \leqq b_1,$$

so that we can put

$$(3) \qquad c_1 + c_2 = a_1 \quad \text{and} \quad c_1 + c_3 = b_1 .$$

[2] Partially ordered semigroups with the refinement property have been discussed in Riesz [1], p. 175. Cf. also Birkhoff [2], pp. 327 ff.; Theorem 13.21 is closely related to Birkhoff's Theorem 49.

We now have by (2) and (3)

$$c_1 + c_3 \leqq c_1 + a_2 \quad \text{and} \quad c_3 \leqq a_2\,;$$

hence, for some c_4,

$$(4) \qquad c_3 + c_4 = a_2\,,$$

and from (1), (3), and (4) we obtain

$$(5) \qquad c_2 + c_4 = b_2\,.$$

Thus (1) implies the existence of elements c_i which satisy (3)–(5); and therefore, by 13.18, $\mathfrak{A}$ has the refinement property. This completes the proof.

DEFINITION 13.22. *A partially ordered algebra $\mathfrak{A} = \langle A, +\rangle$ is called*

(i) *FINITELY COMPLETE (OR LATTICE-ORDERED),*

(ii) *COUNTABLY COMPLETE,*

(iii) *COMPLETE,*

according as the greatest lower bound and the least upper bound exist

(i) *for any elements $a, b \varepsilon A$,*

(ii) *for every infinite sequence of elements $a_i \varepsilon A$,*

(iii) *for any elements $a_i \varepsilon A$ correlated with elements i of an arbitrary non-empty set I.*

DEFINITION 13.23. *A partially ordered algebra $\mathfrak{A} = \langle A, +\rangle$ is called FINITELY COMPLETE IN THE WIDER SENSE if $a \cap b$ exists for any $a, b \varepsilon A$ such that $c \leqq a$ and $c \leqq b$ for some $c \varepsilon A$, and if $a \cup b$ exists for any $a, b \varepsilon A$ such that $a \leqq c$ and $b \leqq c$ for some $c \varepsilon A$. Similarly we define the notion of a COUNTABLY COMPLETE, or COMPLETE, algebra IN THE WIDER SENSE—in analogy to* 13.22(ii),(iii).

THEOREM 13.24. *Every finitely complete semigroup $\mathfrak{A}$ has the refinement property.*[3]

[3] A discussion of finitely and countably complete semigroups is implicitly contained in Birkhoff [2]. Actually, Birkhoff discusses the so-called lattice-ordered groups; however, in view of a result of J. von Neumann quoted loc. cit., p. 306, all his theorems can be automatically applied to finitely complete semigroups. In particular, our Theorem 13.24 and a half of Theorem 13.26 (the fact that conditions (i) and (iii) imply (ii) and (iv)) follow from Birkhoff's results, loc. cit., pp. 321 f. and 328. On the other hand, Theorems 13.12–13.17, which apply to arbitrary partially ordered semigroups, contain extensions of some results of Birkhoff.

PROOF: By 13.22, Theorem 2.28 with $n = p = 2$ obviously holds in $\mathfrak{A}$; hence, by 13.21, the conclusion.

THEOREM 13.25. *Let $\mathfrak{A} = \langle A, + \rangle$ be a finitely complete semigroup; and let a_i and b_i be elements in A correlated with elements i of a non-empty set I. If $a_i + b_i = c$ for every $i \in I$ and if $\bigcap_{i\in I} a_i$ exists, then $\bigcup_{i\in I} b_i$ also exists, and we have*

$$\bigcap_{i\in I} a_i + \bigcup_{i\in I} b_i = c.$$

PROOF: We show first that 3.17 holds for every finitely complete algebra, and that it can be extended even to arbitrary collections of elements a_i; instead of 2.28, we make use of the fact that any two elements of the algebra have a greatest lower bound (cf. 13.22). Hence we derive 13.25 by arguing as in the proof of 4.21.

THEOREM 13.26. *Let $\mathfrak{A} = \langle A, + \rangle$ be a partially ordered semigroup. Each of the following conditions is necessary and sufficient for $\mathfrak{A}$ to be finitely complete or—what amounts to the same thing—to be finitely complete in the wider sense:*

(i) *$a \cap b$ exists for any $a, b \in A$;*

(ii) *$a \cup b$ exists for any $a, b \in A$.*

Similarly, each of the following conditions is necessary and sufficient for $\mathfrak{A}$ to be countably complete in the wider sense:

(iii) *$\bigcap_{i<\infty} a_i$ exists for any $a_0, a_1, \cdots, a_i, \cdots \in A$;*

(iv) *$\bigcup_{i<\infty} b_i$ exists for any $b_0, b_1, \cdots, b_i, \cdots \in A$ provided there is an element $c \in A$ such that $b_i \leqq c$ for $i = 0, 1, 2, \cdots$.*

PROOF: The first part of the theorem follows easily from 13.15, 13.22, and 13.23. As for the second part, we see that the necessity of (iii) follows directly from 13.23 (since $\mathfrak{A}$ has a zero element). Now assume (iii) to hold. By the first part of our theorem, this implies that $\mathfrak{A}$ is finitely complete; and hence 13.25 applies to $\mathfrak{A}$. Consider any elements $b_i, c \in A$ with $b_i \leqq c$ for $i = 0, 1, 2, \cdots$. We have thus

$$a_i + b_i = c$$

for some elements $a_i \in A, i = 0, 1, 2, \cdots$; and since $\bigcap_{i<\infty} a_i$ is assumed to exist, $\bigcup_{i<\infty} b_i$ also exists by 13.25. Thus, (iii) implies (iv). Similarly, assume (iv) to hold, and consider any elements $a_0, a_1, \cdots, a_i, \cdots \in A$. By our assumption and 13.12(ii), $a_0 \cap a_i$ exists for every $i < \infty$, and we can put

$$(a_0 \cap a_i) + b_i = a_0 \quad \text{for} \quad i = 0, 1, 2, \cdots .$$

Hence, again by our assumption and by 13.12(i), $\bigcap_{i<\infty} (a_0 \cap a_i)$ exists, and we obviously have

$$\bigcap_{i<\infty} a_i = \bigcap_{i<\infty} (a_0 \cap a_i).$$

Thus, (iii) holds; and (iii) and (iv) together imply by 13.23 that $\mathfrak{A}$ is countably complete in the wider sense. The necessity and sufficiency of conditions (iii) and (iv) have thus been established.

In connection with this theorem, it is interesting to note that a semigroup $\mathfrak{A}$ is never countably complete, unless it has only one element (for otherwise $\mathfrak{A}$ would have an infinite element $\infty \cdot a$, in contradiction to 13.5).

By modifying in an obvious way conditions (iii) and (iv) in 13.26, we obtain necessary and sufficient conditions for $\mathfrak{A}$ to be complete in the wider sense.

We now turn to the problem of establishing conditions under which a given semigroup is a G.C.A.

THEOREM 13.27. *For a semigroup $\mathfrak{A} = \langle A, + \rangle$ to be a G.C.A., it is necessary and sufficient that $\mathfrak{A}$ be partially ordered, have the refinement property, and satisfy the following condition:*

(i) if $a_0, a_1, \cdots, a_i, \cdots, b \in A$ and $a_n \leqq a_{n+1} \leqq b$ for every $n < \infty$, then $\bigcup_{i<\infty} a_i$ exists.

PROOF: The necessity of the conditions follows from 7.11 and 13.18. Now assume these conditions to be satisfied. We consider the operation $\sum$ on infinite sequences defined in 13.2; our purpose is to show that $\mathfrak{A}' = \langle A, +, \sum \rangle$ is a G.C.A. in the sense of 5.1. Postulates 5.1.I and II hold in $\mathfrak{A}'$ by 13.16 and by condition (i); 5.1.III is obviously satisfied. To derive 5.1.V, consider elements a_i and b_i such that

(1) $$a_n = b_n + a_{n+1} \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

We obtain by induction

$$a_0 = \sum_{i<n} b_i + a_n \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

Hence, by (i), $\sum_{i<\infty} b_i$ exists and is less than or equal to a_0. We therefore have

(2) $$a_0 = c + \sum_{i<\infty} b_i\,;$$

and we conclude from (1) and (2), by means of 5.1.I and the cancellation law for sums, that

$$a_n = c + \sum_{i<\infty} b_{n+i} \quad \text{for} \quad n = 0, 1, 2, \cdots:$$

Thus, c satisfies the conclusion of 5.1.V. Finally, to obtain 5.1.IV, we consider elements $a, b, c_0, c_1, \cdots$ with

$$a + b = \sum_{i<\infty} c_i . \tag{3}$$

We put

$$d_0 = a \quad \text{and} \quad e_0 = b. \tag{4}$$

We then have by 5.1.I

$$d_0 + e_0 = c_0 + \sum_{i<\infty} c_{i+1} ;$$

and by applying the refinement theorem 2.3, we obtain elements a_0, b_0, d_1, and e_1 for which

$$d_0 = a_0 + d_1 , \quad e_0 = b_0 + e_1 , \quad c_0 = a_0 + b_0 ,$$

and

$$d_1 + e_1 = \sum_{i<\infty} c_{i+1} = c_1 + \sum_{i<\infty} c_{i+2} .$$

In view of the last formula, we can continue this procedure indefinitely; and we arrive at four sequences a_i, b_i, d_i, and e_i such that

$$d_n = a_n + d_{n+1} , \quad e_n = b_n + e_{n+1} , \tag{5}$$

$$c_n = a_n + b_n \quad (\text{and} \quad d_n + e_n = \sum_{i<\infty} c_{n+i}) \tag{6}$$

for every $n < \infty$. By applying 5.1.V to (5), we obtain two elements c' and c'' such that, in view of (4),

$$a = c' + \sum_{i<\infty} a_i \quad \text{and} \quad b = c'' + \sum_{i<\infty} b_i . \tag{7}$$

Hence, by (3), (6), and 5.1.II,

$$a + b = \sum_{i<\infty} c_i = c' + c'' + \sum_{i<\infty} (a_i + b_i) = c' + c'' + \sum_{i<\infty} c_i .$$

Consequently, by 13.11(ii),

$$c' + c'' = 0, \quad c' = c'' = 0,$$

and therefore, by (7),

$$a = \sum_{i<\infty} a_i \quad \text{and} \quad b = \sum_{i<\infty} b_i . \tag{8}$$

By (6) and (8) the elements a_i and b_i satisfy the conclusion of 5.1.IV. In view of 5.1 and 13.1, the proof is complete.

THEOREM 13.28. *Every semigroup $\mathfrak{A}$ which is countably complete in the wider sense is a G.C.A.*

PROOF: By 13.23 and 13.26, $\mathfrak{A}$ is partially ordered, finitely complete, and satisfies 13.27(i). Hence, by 13.24, $\mathfrak{A}$ has also the refinement property; and therefore, by 13.27, it is a G.C.A.

Among semigroups which have thus proved to be G.C.A.'s we find various familiar algebraic systems; some of them will be discussed in §14.

The question arises whether the results obtained in the preceding parts of this work have any interesting implications for the semigroups considered in 13.27 and 13.28. In the first place we have in mind here the arithmetical results of Part I. The arithmetic of semigroups is, in general, much simpler than that of C.A.'s. Owing primarily to the cancellation law for sums, many theorems of Part I (like 2.10–2.15) become trivial when applied to semigroups, whereas the proofs of some others (e.g., 2.28) undergo a far-reaching simplification.

The situation is different with regard to the cancellation laws for multiples (2.33 and 2.34) and their consequences (e.g., Euclid's theorem 2.37). A direct proof of these theorems for arbitrary finitely complete semigroups—and hence, in particular, for the semigroups considered in 13.28—is rather simple.[4] We see, however, no possibility of essentially simplifying the involved proof of these results outlined in §2, if we want to apply them to all the semigroups considered in 13.27. It may be interesting to notice that the cancellation laws for multiples do not hold in arbitrary partially ordered semigroups with the refinement property. For instance, the set of all real numbers x with $|x| \geqq 1$ and $x \neq -1$ forms a semigroup under ordinary multiplication; as is easily seen, this semigroup is partially ordered and has the refinement property, but does not satisfy the cancellation laws 2.33 and 2.34 for $m = 2$, since, e.g., $3 \cdot 3 = (-3) \cdot (-3)$, although $3 \neq -3$. This shows that condition 13.27(i) must play an essential part in the derivation of these laws, and seems to indicate the necessity of an infinite construction.

The fact that the cancellation laws for multiples hold in the semigroups which satisfy the conditions of 13.27 has some further con-

[4] 2.33 and 2.34 for finitely complete semigroups have been established in Birkhoff [2], p. 308 (cf. the preceding note). By simplifying the original proof of 2.37, we can show that this theorem holds in every (commutative) semigroup which satisfies 2.33.

sequences. Thus, the smallest group in which every such semigroup can be imbedded is a torsion-free Abelian group, i.e., an Abelian group without elements of a positive finite order. Or, to give another example: If $\mathfrak{A} = \langle A, + \rangle$ is a partially ordered semigroup in which the cancellation law 2.34 holds, then the order established in A by the relation $\leqq$ can always be extended to a simple order; i.e., we can construct a relation $\precsim$ with the following properties:

(i) A is simply ordered by $\precsim$;

(ii) for any $a, b, c \, \varepsilon \, A$, $a \precsim b$ implies $a + c \precsim b + c$;

(iii) for any $a, b \, \varepsilon \, A$, $a \leqq b$ implies $a \precsim b$ (and hence $0 \precsim a$ for every $a \, \varepsilon \, A$).[5]

In view of 14.6, this result automatically applies to all finitely closed G.C.A.'s without infinite elements. By using the properties of finite closures (cf. the end of §7), we can easily extend the result to arbitrary G.C.A.'s without infinite elements, thus, e.g., to all multiple-free G.C.A.'s; of course, we have to assume that the sums involved in (ii) are in A.

From the results obtained in §12 we can derive directly certain conclusions regarding isomorphism types of semigroups. We notice first

THEOREM 13.29. *All the theorems of* §12 *which apply to arbitrary algebras* $\mathfrak{A} = \langle A, +, \sum \rangle$ *apply also to arbitrary algebras* $\mathfrak{A} = \langle A, + \rangle$ (*with a zero element*); *and those which apply to C.A.'s and G.C.A.'s in the sense of* 1.1 *and* 5.1 *apply also to C.A.'s and G.C.A.'s in the sense of* 13.1.

PROOF: The first part of the theorem is obvious. To obtain the second part, we show that the algebra of types of C.A.'s in the sense of 1.1 (or G.C.A.'s in the sense of 5.1) is isomorphic under cardinal multiplication with the algebra of types of C.A.'s (or G.C.A.'s) in the sense of 13.1; the proof is based upon 6.1, 6.11, 6.12, 12.1, 12.2, and 13.1–13.3, and is quite elementary.

In view of 13.7, the fundamental result of §12—Theorem 12.17—finds no direct application in the theory of semigroups. On the other

[5] An analogous result for torsion-free Abelian groups has been obtained by the author and stated (without proof) in Tarski [6]; a proof is outlined in Birkhoff [2], pp. 312 f. The extension of the result to partially ordered semigroups which satisfy 2.34 presents no difficulty.

hand, we can apply Theorem 12.21 to those semigroups which have proved to be G.C.A.'s. We obtain thus:

THEOREM 13.30. T *being the class of types of all semigroups which are countably complete in the wider sense (or which satisfy the hypothesis of* 13.27), $\langle \mathrm{T}, \times \rangle$ *is a finitely closed R.A.*

PROOF: By 13.4 and 13.28 (or 13.27), T is a subset of the types of all finitely closed G.C.A.'s. It is also easily seen that, for any types α and β, $\alpha \times \beta$ is in T if, and only if, both α and β are in T. Hence the conclusion follows by 5.26, 11.26, and 12.21 (13.29).

Further results concerning types of semigroups will be found in the appendix. We shall see, for instance, that the refinement theorem 12.14 applies to types of arbitrary partially ordered semigroups; and that 13.30 can be extended to a wider class of these algebras. We shall also come across a rather comprehensive class of semigroups whose types constitute a C.A.

In conclusion we want to call attention to certain algebras which are in the same relation to groupoids and semigroups as G.C.A.'s to C.A.'s; they may be called GENERALIZED GROUPOIDS and SEMIGROUPS. An algebra $\mathfrak{A} = \langle A, + \rangle$ is a generalized groupoid if it satisfies Postulates 11.26.I, 11.26.II, and 1.1.V; it is a generalized semigroup if, in addition, it satisfies the cancellation law for sums stated in 13.4(ii)—under the assumption that the sums involved are in A. Thus, in particular, all R.A.'s are generalized groupoids. By following the lines of the proof of 7.7 and 7.8, we can show that every generalized groupoid $\mathfrak{A}$ which has the refinement property and in which 1.34 holds (i.e., every algebra satisfying 11.26.I–IV) can be imbedded in a groupoid $\mathfrak{A}'$ with the same properties, which is a finite closure of $\mathfrak{A}$ in the sense of 7.18; if, moreover, $\mathfrak{A}$ is a generalized semigroup, $\mathfrak{A}'$ proves to be a semigroup. We can also show, by arguing as in the proof of 7.14, that the groupoid (or semigroup) $\mathfrak{A}'$ which satisfies these conditions—i.e., which is a finite closure of the generalized groupoid $\mathfrak{A}$,which has the refinement property, and in which 1.34 holds—is uniquely determined by $\mathfrak{A}$ up to isomorphism.

§ 14. ALGEBRAS OF INTEGERS AND REAL NUMBERS

Theorems 13.27 and 13.28 of the preceding section provide us with various simple examples of G.C.A.'s. We have, for example,

THEOREM 14.1. *I being the set of all finite non-negative integers, and N the set of all finite non-negative real numbers, the algebras* $\mathfrak{J} = \langle I, + \rangle$ *and* $\mathfrak{N} = \langle N, + \rangle$ *(where + is ordinary arithmetical addition) are finitely closed G.C.A.'s.*

PROOF: by 13.23 and 13.28.

Hence we obtain further examples by applying general methods of construction which have been discussed in §6. Thus, the algebra $\mathfrak{N}$ of 14.1 has a great variety of generalized cardinal subalgebras; one of them is, of course, the algebra $\mathfrak{J}$. The strong cardinal powers of $\mathfrak{N}$ are algebras constituted by all functions f, with a given domain D, which assume non-negative real numbers as values; in particular, the 'cardinal square' of $\mathfrak{N}$ is the G.C.A. of all complex numbers $r + s \cdot i$ with $r \geqq 0$ and $s \geqq 0$. As an isomorphic image of $\mathfrak{N}$ we obtain the algebra of all real numbers $r \geqq 1$ under ordinary multiplication; and among generalized cardinal subalgebras of the latter we find the multiplicative algebra of positive finite integers, which is isomorphic with a weak cardinal power of $\mathfrak{J}$.

The construction of C.A.'s which are closures of $\mathfrak{J}$ and $\mathfrak{N}$ is very simple; we include the infinite number ∞ in the sets I and N by stipulating, as usual, that

$$r + \infty = \infty + r = \infty + \infty = \infty$$

for every non-negative number r. In fact, we have the following:

THEOREM 14.2. $\bar{I}$ *being the set of all non-negative integers, and* $\bar{N}$ *the set of all non-negative real numbers, both with* ∞ *included, the algebras* $\bar{\mathfrak{J}} = \langle \bar{I}, + \rangle$ *and* $\bar{\mathfrak{N}} = \langle \bar{N}, + \rangle$ *(where + is ordinary arithmetical addition) are C.A.'s, and are closures of the algebras* $\mathfrak{J}$ *and* $\mathfrak{N}$ *of* 14.1, *respectively.*

PROOF: It is easy to verify directly that the algebras $\langle \bar{I}, + \rangle$ and $\langle \bar{N}, + \rangle$ when supplemented with the operation $\sum$ of 13.2 satisfy all postulates of 1.1 and all conditions of 7.1. The following argu-

ment, however, is perhaps somewhat shorter and more interesting. Consider, e.g., the G.C.A. $\langle N, +\rangle$ of 14.1. By 7.8 and 13.1 there is a C.A. $\langle N', +'\rangle$ which is a closure of $\langle N, +\rangle$. From 9.32 we conclude that $\langle N, +\rangle$ is simple; hence, by 10.15, $\langle N', +'\rangle$ is also simple. Therefore, by 4.12 and 9.34, N' has just one infinite element ∞'; by 4.4 and 9.33, we have

$$r +' \infty' = \infty' +' r = \infty' +' \infty' = \infty' \quad \text{for every} \quad r \;\varepsilon\; N'.$$

From 7.1(iii) and 13.3 we see that every element $r \;\varepsilon\; N'$ is the least upper bound of a sequence of elements $s_i \;\varepsilon\; N$; and we easily conclude (e.g., with the help of 2.22, 4.12, and 4.14) that if r is not itself in N, it must be infinite, and hence equal to ∞'. We can now construct a function f which (according to 6.1) maps $\langle N', +'\rangle$ isomorphically onto $\langle \bar{N}, +\rangle$; we simply put

$$f(r) = r \quad \text{for every} \quad r \;\varepsilon\; N, \quad \text{and} \quad f(\infty') = \infty.$$

Consequently, $\langle \bar{N}, +\rangle$ is a C.A. and a closure of $\langle N, +\rangle$.

The C.A.'s $\bar{\mathfrak{J}}$ and $\bar{\mathfrak{R}}$ of 14.2 have certain simple, but rather interesting, algebraic properties, which will be discussed in the next few theorems. We want to show first that every C.A. which is not idem-multiple contains a subalgebra isomorphic with $\bar{\mathfrak{J}}$.

THEOREM 14.3. *Let $\mathfrak{A} = \langle A, +\rangle$ be a C.A.; let a be an element in A, and let B be the set of elements $n \cdot a$ with $n \leqq \infty$. Then $\mathfrak{B} = \langle B, +\rangle$ is a cardinal subalgebra of $\mathfrak{A}$; and, if a is not idem-multiple, $\mathfrak{B}$ is isomorphic with the algebra $\bar{\mathfrak{J}}$ of 14.2. (If $a = 0$, B consists only of 0; and if a is idem-multiple but $\neq 0$, then B consists of two elements, 0 and a, and in this case $\mathfrak{B}' = \langle B', +\rangle$ where B' consists only of a is also a cardinal subalgebra of $\mathfrak{A}$.)*

PROOF: If a is not idem-multiple, we have by 4.3

$$n \cdot a = p \cdot a + q \cdot a \quad \text{if, and only if,} \quad n = p + q$$

for any integers n, p, and q. Hence, by 6.2, 6.13, and 14.2, $\mathfrak{B}$ is isomorphic with $\bar{\mathfrak{J}}$ and is a cardinal subalgebra of $\mathfrak{A}$. (If a is idem-multiple, the proof is obvious; we apply 4.3 again.)

The question arises under what conditions a C.A. $\mathfrak{A}$ contains a subalgebra $\mathfrak{B}$ which is isomorphic with the algebra $\bar{\mathfrak{R}}$ of real numbers. In this connection the following result may be mentioned:

Let $\mathfrak{A} = \langle A, +\rangle$ be a C.A.; let a be an element in A such that all multiples $r \cdot a$ with a real $r \geqq 0$ exist. If a is not idem-multiple, and

if B is the set of all these multiples $r \cdot a$, then $\mathfrak{B} = \langle B, +\rangle$ is a cardinal subalgebra of $\mathfrak{A}$ and is isomorphic with the algebra $\overline{\mathfrak{R}}$ of 14.2.

The proof will not be given here; regarding the notion of a real multiple and conditions for the existence of such multiples, compare the closing remarks of §2.

THEOREM 14.4. *Each of the following two conditions is necessary and sufficient for a C.A. $\mathfrak{A} = \langle A, +\rangle$ to be isomorphic with the C.A. $\overline{\mathfrak{J}}$ of* 14.2:

(i) *A has no two different infinite elements (i.e., $\mathfrak{A}$ is simple) and has an indecomposable element;*

(ii) *there is an element $a \in A$ such that a is not idem-multiple, and A consists only of elements $n \cdot a$ with $n \leqq \infty$.*

PROOF: The necessity of (i) is obvious (cf. 4.38 and 9.34); by 4.39 and 9.35, (i) implies (ii); and the sufficiency of (ii) follows from 14.3.

THEOREM 14.5. *The following conditions are severally necessary and jointly sufficient for a C.A. $\mathfrak{A} = \langle A, +\rangle$ to be isomorphic with the C.A. $\overline{\mathfrak{R}}$ of* 14.2:

(i) *A has no two different infinite elements (i.e., $\mathfrak{A}$ is simple);*

(ii) *A has no indecomposable elements;*

(iii) *A has at least one finite element besides* 0;

(iv) *A is simply ordered by the relation $\leqq$ (i.e., $\mathfrak{A}$ is simply ordered).*

PROOF: The necessity of conditions (i)–(iv) is obvious (cf. 4.10, 4.38, 9.34, and 13.10). Now assume these conditions to be satisfied. Let B be the set of all finite elements of A. From (ii)–(iv), 4.14, 4.38, 7.11, and 13.8 it is easily seen that the algebra $\mathfrak{B} = \langle B, +\rangle$ has the following properties: it is a commutative semigroup which is simply ordered, has at least two different elements, has no smallest element $\neq 0$, and in which every increasing sequence of elements in B which is bounded above has a least upper bound. It is known, however, that these properties characterize up to isomorphism the G.C.A. $\mathfrak{R}$ of 14.1;[6] hence, $\mathfrak{B}$ and $\mathfrak{R}$ are isomorphic, and $\mathfrak{B}$ is a G.C.A. If $a \neq 0$ is an element of B, then by (i) and 4.12, $\infty \cdot a$ is the only element in A which is not in B. Hence, by 7.1, we easily conclude that $\mathfrak{A}$ is a closure of $\mathfrak{B}$, while, by 14.2, $\overline{\mathfrak{R}}$ is a closure of $\mathfrak{R}$. By applying 7.14 now, we obtain

$$\mathfrak{A} \cong \overline{\mathfrak{R}}.$$

[6] Cf., e.g., Huntington [1], p. 277.

This completes the proof.

THEOREM 14.6. *The following algebras $\mathfrak{P}$ are the only cardinal subalgebras of the algebra $\overline{\mathfrak{R}}$ of* 14.2 (*i.e., the only C.A.'s constituted by a set of non-negative real numbers under ordinary addition*):

(i) *the algebra $\overline{\mathfrak{R}}$ itself;*

(ii) *the algebras $\langle P, +\rangle$ where P is the set of all multiples $n \cdot r$ of a given real number r, $0 < r < \infty$ (n being an integer $\leqq \infty$);*

(iii) *the algebras $\langle P, +\rangle$ where P consists only of* 0 *or only of ∞ or of both these numbers.*

PROOF: By 6.13, 14.2, and 14.3, all algebras $\mathfrak{P}$ listed above are cardinal subalgebras of $\overline{\mathfrak{R}}$. Now let $\mathfrak{P} = \langle P, +\rangle$ be any cardinal subalgebra of $\overline{\mathfrak{R}}$; thus, P is a set of non-negative real numbers. If P contains no finite number $r \neq 0$, $\mathfrak{P}$ obviously coincides with one of the algebras listed in (iii). If P has a smallest finite number $r \neq 0$, then, by 4.38, r is indecomposable in $\mathfrak{P}$, and hence, by 14.4, $\mathfrak{P}$ must be one of the algebras listed in (ii). If, finally, P contains a finite number $r \neq 0$, but has no smallest such number, there must be an infinite sequence of numbers $r_0, r_1, \cdots, r_i, \cdots \varepsilon P$ which are different from 0, but which approach 0. As is easily seen, every real number $x \geqq 0$ can then be represented in the form

$$x = \sum_{i<\infty} n_i \cdot r_i$$

where $n_0, n_1, \cdots, n_i, \cdots$ are integers $\leqq \infty$; and a simple argument shows that the operations $\sum$ in $\mathfrak{P}$ and $\overline{\mathfrak{R}}$ coincide. Hence, P is the set of all real numbers (with ∞ included), and $\mathfrak{P} = \overline{\mathfrak{R}}$. The proof is thus complete.

THEOREM 14.7. *For a C.A. $\mathfrak{A} = \langle A, +\rangle$ to be isomorphic with one of the algebras $\mathfrak{P}$ of* 14.6 *it is necessary and sufficient that $\mathfrak{A}$ satisfy conditions* (i) *and* (iv) *of* 14.5 (*i.e., that $\mathfrak{A}$ be simple and simply ordered*).

PROOF: The necessity is obvious. Now let $\mathfrak{A} = \langle A, +\rangle$ be a C.A. which satisfies 14.5(i),(iv). If A has no finite element $\neq 0$, then A has at most two different elements, and the proof that $\mathfrak{A}$ is isomorphic with one of the algebras $\langle P, +\rangle$ of 14.6(iii) is obvious. Otherwise we apply either 14.4(i) or 14.5—depending on whether or not A has an indecomposable element.

THEOREM 14.8. *Theorems* 14.5 *and* 14.7 *remain valid if condition* 14.5(iv) *is replaced by any one of the following conditions:*

(i) *$a \cap b$ exists for any $a, b \;\varepsilon\; A$;*

(ii) *$a \cup b$ exists for any $a, b \;\varepsilon\; A$;*

(iii) *every subset B of A which is well ordered by the relation $\leqq$ is at most denumerable;*

(iv) *every subset B of A which is inversely well ordered by the relation $\leqq$ (i.e., which is well ordered by the relation $\geqq$) is at most denumerable.*

PROOF: By 9.39, conditions 14.5(iv), 14.8(i), and 14.8(ii) are equivalent for every C.A. $\mathfrak{A}$ which satisfies 14.5(i). Moreover, we see from 14.7 that every C.A. $\mathfrak{A}$ which satisfies 14.5(i),(iv) satisfies 14.8(iii),(iv) as well. Finally, by 3.35 and 3.36, condition 14.8(iii) implies 14.8(i), and condition 14.8(iv) implies 14.8(ii).

The question arises whether condition (iv) in 14.5 and 14.7 (or the equivalent conditions in 14.8) can be entirely omitted. This is a problem which was mentioned already at the end of §9—that of the existence of a simple C.A. which is not simply ordered; now, however, it assumes the form of a problem concerning an axiomatic characterization of the algebra of real numbers.

We are now going to show that the C.A.'s of 14.6 and 14.7 are simple not only in the sense of 9.31 but also in another, probably stronger, sense: they have no coset algebras which are C.A.'s, except the 'trivial' coset algebras mentioned in §6 in connection with 6.10. It is easily seen from 9.30 that every C.A. which is simple in the new sense is also simple in the old sense; the problem, however, remains open whether the converse also holds.

THEOREM 14.9. *Let $\mathfrak{P} = \langle P, +\rangle$ be a C.A. constituted by a set P of non-negative real numbers under ordinary addition. The only infinitely additive equivalence relations R which generate new C.A.'s $\mathfrak{P}/R$ are: the universal relation ($x \mathrel{R} y$ for any $x, y \in P$), the identity relation ($x \mathrel{R} y$ if, and only if, $x = y$), and the relation which holds between two elements $x, y \in P$ if, and only if, either $x = y = 0$ or else $x \neq 0$ and $y \neq 0$.*

PROOF: Assume R to be different from the identity relation. We then have, in view of 14.6,

$$(r + s) \mathrel{R} s \quad \text{for some} \quad r, s \in P \text{ with } r \neq 0.$$

Hence, by 7.13(iii),

(1) $$(\infty \cdot r + s) \mathrel{R} s, \quad \text{i.e.,} \quad \infty \mathrel{R} s.$$

Now if x and y are any numbers in P different from 0, we have

(2) $$s \leqq n \cdot x \leqq \infty \quad \text{and} \quad s \leqq n \cdot y \leqq \infty \quad \text{for some } n, \quad 0 < n < \infty.$$

(1) and (2) imply by 7.13(ii)

$$(n \cdot x) \; R \; (n \cdot y),$$

and consequently, by 7.13(v),

$$x \, R \, y.$$

Hence we see that R is either the universal relation, or else the relation which holds between $x, y \; \varepsilon \; P$ if, and only if, $x = y = 0$ or $x \neq 0$ and $y \neq 0$. This completes the proof.

Theorems 14.3–14.5 can be slightly modified so as to apply to algebras $\mathfrak{J}$ and $\mathfrak{R}$ of 14.1; for this purpose it suffices to consider finitely closed G.C.A.'s instead of C.A.'s, to exclude all infinite elements, and to stipulate in 14.4(i) that A has just one indecomposable element. On the other hand, Theorems 14.6–14.9, with the changes just indicated, do not apply to arbitrary subalgebras $\langle P, + \rangle$ of $\mathfrak{R}$, but only to a rather special class of such subalgebras, in fact, to those in which P, together with any numbers r and s, contains not only $r + s$ but also $| r - s |$. The algebra $\mathfrak{R}$ has also, however, many other generalized cardinal subalgebras which cannot be characterized in the way indicated in 14.6 or 14.7 and which, in particular, are not simple. As an example we can give the subalgebra constituted by all numbers $n + p \cdot \sqrt{2}$ where n and p are finite non-negative integers. It is interesting to notice that if we include ∞ in this subalgebra, we obtain, not a C.A., but 'almost' a C.A.; in fact, an algebra which satisfies Postulates 1.1.I–V and 1.1.VII as well as the refinement theorem 2.3, but not the refinement postulate 1.1.VI. An analogous remark applies, e.g., to the multiplicative algebra of positive integers previously mentioned.

We have considered so far subalgebras, isomorphic images, and coset algebras of the algebras of integers and real numbers. As regards cardinal products, we state the following rather obvious theorem applying to the algebra of integers:

Theorem 14.10. *For every algebra* $\mathfrak{A} = \langle A, + \rangle$ *the following two conditions are equivalent:*

(i) *there is a set* B *such that* $\mathfrak{A} \cong \overline{\mathfrak{J}}^B$ *where* $\overline{\mathfrak{J}}$ *is the algebra of integers of* 14.2;

(ii) $\mathfrak{A}$ *is a C.A., and every element* $a \; \varepsilon \; A$ *can be represented in the form*

$$a = \sum_{i<n} a_i$$

where $n \leqq \infty$, *and the elements* a_i *with* $i < n$ *are indecomposable. If these conditions are satisfied, we can take for* B *in* (i) *the set of all indecomposable elements in* A.

PROOF: The algebra $\bar{\mathfrak{J}}$ of 14.2 obviously satisfies (ii), the number 1 being by 4.38 the (only) indecomposable element of $\bar{\mathfrak{J}}$; hence, by 6.1 and 6.11 we conclude that (i) implies (ii). Now assume (ii) to hold, and let B be the set of all indecomposable elements of A. By 1.42 and 4.45, every element $a \varepsilon A$ can be uniquely represented in the form

$$(1) \qquad a = \sum_{i<n} m_i \cdot b_i$$

where $n \leqq \infty$, $m_0, m_1, \cdots$ are integers $\neq 0$, and $b_0, b_1, \cdots$ are distinct elements of B. Hence, we can correlate with every a in A a function f_a by putting

$$f_a(b) = m \quad \text{if } b \text{ occurs in (1) with the coefficient } m,$$

and

$$f_a(b) = 0 \quad \text{if } b \varepsilon B \text{ and } b \text{ does not occur in (1).}$$

From 6.1 and 6.11 we easily see that f maps $\mathfrak{A}$ isomorphically onto $\bar{\mathfrak{J}}^B$, which completes the proof.

A similar theorem applies to the algebra $\mathfrak{J}$ of 14.1:

THEOREM 14.11. *For every algebra* $\mathfrak{A} = \langle A, + \rangle$ *the following three conditions are equivalent:*

(i) *there is a set* B *such that* $\mathfrak{A} \cong \mathfrak{J}_w^B$ *where* $\mathfrak{J}$ *is the algebra of finite integers of* 14.1;

(ii) $\mathfrak{A}$ *is a finitely closed G.C.A., and every element* $a \varepsilon A$ *can be represented in the form*

$$a = \sum_{i<n} a_i$$

where $n < \infty$ *and the elements* a_i *with* $i < n$ *are indecomposable;*

(iii) $\mathfrak{A}$ *is a groupoid, and every element* $a \varepsilon A$ *has, apart from order, a unique representation described in* (ii); *the term 'groupoid' can be replaced here by 'semigroup.'*

If these conditions are satisfied, we can take for B *in* (i) *the set of all indecomposable elements in* A.

PROOF: By arguing as in the preceding proof, we show that (i) implies (ii), and (iii) implies (i); (ii) implies (iii) by 4.45. Finally, it is easily seen that the unicity of representation in (ii) implies the cancellation law for sums; hence, we can replace in (iii) 'groupoid' by 'semigroup.' This completes the proof.

Conditions (ii) and (iii) of this theorem can be modified in several ways. Thus, the condition that every element a in a G.C.A. or a groupoid has a representation described in (ii) can be replaced by the FINITE CHAIN CONDITION:

If $a_0, a_1, \cdots, a_i, \cdots, b_0, b_1, \cdots, b_i, \cdots \varepsilon A$ and $a_n = b_n + a_{n+1}$ for $n = 0, 1, 2, \cdots$, then there is an integer $m < \infty$ such that $b_{m+i} = 0$ for $i = 0, 1, 2, \cdots$.

When applied to G.C.A.'s, this condition can also be expressed in the following way:

If $a_0, a_1, \cdots, a_i, \cdots$ and $\sum_{i<\infty} a_i$ are in A, then there is an integer m such that $a_{m+i} = 0$ for $i = 0, 1, 2, \cdots$.

On the other hand, the condition that every element a in a groupoid has at most one representation of the kind considered can be replaced by the condition that the groupoid has the refinement property (in the sense of 13.18).

An important notion applying to arbitrary algebras but essentially involving real numbers is that of MEASURE.[7]

DEFINITION 14.12. *Let $\mathfrak{A} = \langle A, + \rangle$, and let u be any element of A. By a (FINITELY ADDITIVE) MEASURE IN $\mathfrak{A}$ NORMED BY u (or WITH THE UNIT u) we understand a function f satisfying the following conditions:*

(i) *$D(f) = A$ and $C(f)$ is a set of real numbers x with $0 \leqq x \leqq \infty$;*
(ii) *if $a, b, a + b \varepsilon A$, then $f(a + b) = f(a) + f(b)$;*
(iii) *$f(u) = 1$.*

Thus, in case $\mathfrak{A}$ is finitely closed, we can say that a measure is a function f which maps $\mathfrak{A}$ homomorphically onto a subalgebra $\mathfrak{P}$ of the algebra $\overline{\mathfrak{R}}$ of 14.2, and correlates the number 1 with a given element u of $\mathfrak{A}$.

[7] In connection with the following discussion, and in particular for the proof of 14.13, see Tarski [1].

The problem arises of establishing simple and 'workable' criteria (i.e., necessary and sufficient conditions, or even only sufficient conditions) for the existence of a measure in any given algebra $\mathfrak{A}$. The solution of this problem assumes a simple form if we restrict ourselves to the case when $\mathfrak{A}$ is a groupoid:

THEOREM 14.13. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be a groupoid, and let* u *be any element in* A. *In order that there exist a finitely additive measure in* $\mathfrak{A}$ *normed by* u *it is necessary and sufficient that* u *satisfy the following condition:*

(i) $(n + 1) \cdot u \leqq n \cdot u$ *does not hold for any* $n < \infty$.

In case $\mathfrak{A}$ *is a partially ordered groupoid, this condition can be replaced by*

(ii) $n \cdot u \neq (n + 1) \cdot u$ *for every* $n < \infty$.

The proof of this theorem can be found in the literature and will not be given here.[8] The necessity of the conditions is obvious, and the second part of the theorem follows obviously from the first.

COROLLARY 14.14. *If* $\mathfrak{A} = \langle A, + \rangle$ *is a partially ordered groupoid in which* 2.12 *or* 2.34 *holds, then the condition* (ii) *of* 14.13 *can be replaced by*

(iii) $u \neq 2 \cdot u$.

This also applies to the case when $\mathfrak{A}$ *is an arbitrary G.C.A.*

PROOF: 14.13(ii) obviously implies (iii). The implication in the opposite direction under the assumption that 2.12 holds is also obvious. From 2.34 we can easily derive 2.12; if, in fact, for any given $a, b, c \in A$,

$$a + n \cdot c = b + n \cdot c,$$

we obtain by induction

$$m \cdot a + n \cdot c = m \cdot b + n \cdot c \quad \text{for every} \quad m < \infty \ (m \neq 0 \text{ if } n \neq 0),$$

and hence, by taking $m = n$ and by applying 2.34,

$$a + c = b + c.$$

The result obviously applies to arbitrary C.A.'s; and it can be extended to G.C.A.'s by 7.8.

COROLLARY 14.15. *If* $\mathfrak{A} = \langle A, + \rangle$ *is a partially ordered semigroup, then the condition* (ii) *of* 14.13 *can be replaced by*

(iv) $u \neq 0$.

[8] See the preceding footnote.

The result also applies to the case when $\mathfrak{A}$ is an arbitrary G.C.A. in which every element is finite.

PROOF: The result is obvious in case $\mathfrak{A}$ is a partially ordered semigroup; hence, by 13.6, it applies to the case when $\mathfrak{A}$ is a finitely closed G.C.A. in which every element is finite; and, with the help of 7.6, 7.8, and 13.8, we easily show that the restriction to finitely closed G.C.A.'s is not essential.

A measure f in an algebra $\mathfrak{A} = \langle A, + \rangle$ normed by an element $u \in A$ is called INFINITELY ADDITIVE if, in addition to 14.12(i)–(iii), it satisfies the condition:

(iv) *if* $a_0, a_1, \cdots, a_i, \cdots$, *and* $\sum_{i<\infty} a_i$ *are in* A, *then*

$$f\,(\sum_{i<\infty} a_i) = \sum_{i<\infty} f(a_i).$$

It is called STRICTLY POSITIVE if

(v) $f(a) = 0$ *implies* $a = 0$ *for every* $a \in A$.

The problem of establishing general criteria for the existence of an infinitely additive measure, or of a measure which is strictly positive and infinitely—or even only finitely—additive, remains open.

§ 15. LATTICES AND BOOLEAN ALGEBRAS

A LATTICE can be regarded as a system formed by a non-empty set A and a binary relation $\leqq$; the relation $\leqq$ is assumed to establish in A a partial order, and the bounds $a \cap b$ and $a \cup b$ are assumed to exist for any elements a and b in A.[9] For our purposes, however, it is more convenient to treat lattices as algebras constituted by a set and a binary operation; moreover, we restrict ourselves to the consideration of lattices with a zero element.

DEFINITION 15.1. *An algebra* $\mathfrak{A} = \langle A, + \rangle$ *is called a* LATTICE *if it is an idem-multiple and finitely complete groupoid.*

It will be seen from Theorem 15.3 that the condition of finite completeness in 15.1 can be replaced by a weaker condition.

THEOREM 15.2. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be an idem-multiple groupoid or, in particular, a lattice. We then have for any elements* $a, b, a_0, a_1, \cdots, a_i, \cdots \varepsilon A$:

(i) $a \leqq b$ *if, and only if,* $a + b = b$;

(ii) $a \cup b = a + b$;

(iii) $\bigcup_{i<n} a_i = \sum_{i<n} a_i$ *in case* $n < \infty$.

The last formula also holds in case $n = \infty$, *under the assumption that* $\bigcup_{i<n} a_i$ *or* $\sum_{i<n} a_i$ *exists.*

PROOF: We easily obtain (i) by 8.1; (ii) follows from (i) by 3.2. In case $n < \infty$, we derive (iii) from (ii) by induction; and by 14.2 the result can be extended to the case when $n = \infty$.

THEOREM 15.3. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be an idem-multiple groupoid. Then*

(i) $\mathfrak{A}$ *is partially ordered;*

(ii) *for* $\mathfrak{A}$ *to be a lattice it is sufficient and necessary that* $a \cap b$ *exist for any* $a, b \varepsilon A$.

PROOF: by 13.10, 13.22, 15.1, and 15.2(i),(ii).

We distinguish various kinds of lattices. In view of 13.22 and 13.23 we can speak of countably complete and of complete lattices (in the strict sense and in the wider sense). Furthermore, we define:

[9] In connection with this section consult Birkhoff [3].

DEFINITION 15.4. *Let $\mathfrak{A} = \langle A, + \rangle$ be a partially ordered algebra.*

(i) *$\mathfrak{A}$ is called* FINITELY DISTRIBUTIVE *if Theorems 3.30 and 3.32 with $n < \infty$ hold for any elements in A.*

(ii) *$\mathfrak{A}$ is called* COUNTABLY DISTRIBUTIVE *if these theorems with $n \leqq \infty$ hold for any elements in A.*

(iii) *$\mathfrak{A}$ is called* COMPLETELY DISTRIBUTIVE *if analogous laws apply to every element $a \in A$, and to any elements $b_i \in A$ correlated with elements i of an arbitrary set I.*

If we apply this definition to algebras which are finitely, countably, or simply complete, the existential assumptions in the distributive laws 3.30 and 3.32 can, of course, be partially or entirely removed. Thus, for instance, a lattice $\mathfrak{A} = \langle A, + \rangle$ is completely distributive according to 15.4 if, for every element $a \in A$ and for any elements $b_i \in A$ correlated with elements i of an arbitrary set I, the existence of $\bigcap_{i \in I} b_i$, or $\bigcup_{i \in I} b_i$, implies that $\bigcap_{i \in I} (a \cup b_i)$, or $\bigcup_{i \in I} (a \cap b_i)$, also exists and that

$$a \cup \bigcap_{i \in I} b_i = \bigcap_{i \in I} (a \cup b_i), \quad \text{or} \quad a \cap \bigcup_{i \in I} b_i = \bigcup_{i \in I} (a \cap b_i).$$

THEOREM 15.5. *For every lattice $\mathfrak{A} = \langle A, + \rangle$ the following conditions are equivalent:*

(i) *$\mathfrak{A}$ is finitely distributive;*

(ii) *Theorems 3.30 with $n = 2$ holds for any elements of A;*

(iii) *Theorem 3.32 with $n = 2$ holds for any elements of A;*

(iv) *$\mathfrak{A}$ has the refinement property.*

PROOF: The equivalence of (i)–(iii) is a well known fact; it applies to arbitrary finitely complete algebras. Now assume conditions (i)–(iii) to hold, and consider any elements $a_1, a_2, b_1, b_2 \in A$ which satisfy the hypothesis of 2.3, i.e.,

$$a_1 + a_2 = b_1 + b_2 .$$

We can then easily show that the elements

$$c_1 = a_1 \cap b_1 , \quad c_2 = a_1 \cap b_2 , \quad c_3 = a_2 \cap b_1 , \quad \text{and} \quad c_4 = a_2 \cap b_2 ,$$

whose existence follows from 13.22 and 15.1, satisfy the conclusion of 2.3, and that consequently (iv) holds (cf. 13.18). In fact, by (ii), 15.2(ii), and 15.3, we have, e.g.,

$$a_1 = a_1 \cap (b_1 \cup b_2) = (a_1 \cap b_1) \cup (a_1 \cap b_2) = c_1 \cup c_2 = c_1 + c_2 .$$

Now assume (iv) to be satisfied. Thus, by 13.18, 2.3 holds in $\mathfrak{A}$; and hence we can derive 2.4 and 3.3. For any elements a, b, and c in A we obviously have

$$a \cap (b + c) \leqq a \quad \text{and} \quad a \cap (b + c) \leqq b + c;$$

hence by 3.3 and 15.2(ii)

$$a \cap (b \cup c) \leqq (a \cap b) \cup (a \cap c).$$

The inequality in the opposite direction follows from 3.1 and 3.2; so that finally we obtain 3.32 with $n = 2$, i.e., our condition (iii). This completes the proof.

In the next few theorems we are going to establish certain connections between C.A.'s (or G.C.A.'s) and lattices. The theorems have a very elementary character.

THEOREM 15.6. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be a G.C.A.*

(i) *For* $\mathfrak{A}$ *to be a lattice it is necessary and sufficient that* $\mathfrak{A}$ *be idem-multiple and finitely closed, and that* $a \cap b$ *exist for any* $a, b \in A$.

(ii) *For* $\mathfrak{A}$ *to be a countably complete lattice it is necessary and sufficient that* $\mathfrak{A}$ *be an idem-multiple C.A., and that* $\bigcap_{i<\infty} a_i$ *exist for any* $a_0, a_1, \cdots, a_i, \cdots \in A$.

(iii) *For* $\mathfrak{A}$ *to be a countably complete lattice in the wider sense it is necessary and sufficient that* $\mathfrak{A}$ *be idem-multiple and finitely* **closed**, *and that* $\bigcap_{i<\infty} a_i$ *exist for any* $a_0, a_1, \cdots, a_i, \cdots \in A$.

PROOF: (i) by 13.6(i), 15.1, 15.3; (ii) and (iii) by (i), 4.7, 7.12, 8.1, 13.22, 13.23, and 15.2(iii).

THEOREM 15.7. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be an idem-multiple, at most denumerable G.C.A.; or, more generally, an idem-multiple G.C.A. in which every subset* B *of* A *that is well ordered by the relation* $\leqq$ *is at most denumerable. If* $\mathfrak{A}$ *is finitely closed, then it is a countably complete lattice in the wider sense; and if* $\mathfrak{A}$ *is a C.A., it is a countably complete lattice.*

PROOF: by 3.35 and 15.6.

THEOREM 15.8. *Every finite and finitely closed G.C.A.* $\mathfrak{A} = \langle A, + \rangle$ *is a C.A. and a complete lattice.*

PROOF: The finiteness of A has the following consequences: First,

for every infinite sequence of elements $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$, there must be a number m such that

$$\sum_{i<m} a_i = \sum_{i<n} a_i \quad \text{for} \quad m \leqq n < \infty,$$

and that consequently, by 7.10,

$$\sum_{i<m} a_i = \sum_{i<\infty} a_i.$$

Thus, $\mathfrak{A}$ is a C.A. Secondly, by 4.3 (or 14.3) and 8.1, $\mathfrak{A}$ must be idem-multiple. Hence, by 15.7, the conclusion.

THEOREM 15.9. *For a lattice $\mathfrak{A} = \langle A, + \rangle$ to be a G.C.A. it is necessary and sufficient that $\mathfrak{A}$ be countably distributive and satisfy the infinite chain postulate* 5.1.V; *instead of requiring that $\mathfrak{A}$ be countably distributive it suffices to require that $\mathfrak{A}$ satisfy the distributive law* 3.32.

PROOF: The necessity of the conditions follows obviously from 3.30, 3.32, and 15.4(ii). Now assume $\mathfrak{A}$ to satisfy 3.32 and 5.1.V. Given any infinite sequence of elements $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$, with a common upper bound b, we have by 15.2(i)

$$a_n + b = b \quad \text{for} \quad n = 0, 1, 2, \cdots;$$

and we conclude by 5.1.V that $\sum_{i<\infty} a_i$ is in A. Hence, by applying 15.2(ii),(iii) and elementary properties of the least upper bound, we easily derive 5.1.I, II; 5.1.III obviously holds in $\mathfrak{A}$; and 5.1.IV is a consequence of the distributive law 3.32 (cf. the proof of 15.5). Thus, $\langle A, +, \sum \rangle$, and therefore also $\mathfrak{A} = \langle A, + \rangle$, is a G.C.A.

THEOREM 15.10. *Every countably complete and countably distributive lattice $\mathfrak{A} = \langle A, + \rangle$ is a C.A.; every lattice $\mathfrak{A}$ which is countably complete in the wider sense and countably distributive is a G.C.A.*

PROOF: By 15.2(ii),(iii), the operations $+$ and $\sum$ coincide in $\mathfrak{A}$ with $\cup$ and $\bigcup$. By remembering this and by reasoning exactly as in the proof of 8.10, we show that $\mathfrak{A}$ satisfies 5.1.V; we apply here 13.23 (or 13.22) and 15.4. Hence, by 15.9, $\mathfrak{A}$ is a G.C.A.; and, in case $\mathfrak{A}$ is countably complete (in the strict sense), it is a C.A. by 13.22 and 15.2(iii).

COROLLARY 15.11. *Every finite and finitely distributive lattice $\mathfrak{A}$ is a C.A.*

PROOF: By 13.22, 15.1, 15.4, and 15.10.

THEOREM 15.12. *Let $\mathfrak{A} = \langle A, + \rangle$ be a lattice. For $\mathfrak{A}$ to be a C.A., it is necessary that:*

(i) *$\bigcup_{\iota<\infty} a_\iota$ exist for any $a_0, a_1, \cdots, a_\iota, \cdots \varepsilon A$.*

For $\mathfrak{A}$ to be a G.C.A., it is necessary that

(ii) *$\bigcup_{\iota<\infty} a_\iota$ exist for any $a_0, a_1, \cdots, a_\iota, \cdots \varepsilon A$ provided there is an element $b \varepsilon A$ such that $a_\iota \leqq b$ for $i = 0, 1, 2, \cdots$.*

In case $\mathfrak{A}$ is simply ordered, these conditions are also sufficient.

PROOF: The necessity of conditions (i) and (ii) follows at once from 8.2, 8.3, and 15.1. Now assume $\mathfrak{A}$ to satisfy (ii) and to be simply ordered. As is easily seen from 15.4, the latter condition implies that $\mathfrak{A}$ is countably (and even completely) distributive. To show that 5.1.V holds in $\mathfrak{A}$, consider elements $a_0, a_1, \cdots, a_\iota, \cdots, b_0, b_1, \cdots, b_i, \cdots \varepsilon A$ such that

$$(1) \qquad a_n = b_n + a_{n+1} \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

Hence, by 15.2(ii) and in view of the fact that $\mathfrak{A}$ is simply ordered, we have

$$(2) \qquad a_n = b_n \quad \text{or} \quad a_n = a_{n+1} \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

Now it may happen that the element $c = 0$ satisfies the conclusion of 5.1.V, i.e., by 15.2(iii),

$$a_n = \bigcup_{i<\infty} b_{n+i} \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

If this is not the case, then we have by (1), for some $m < \infty$,

$$a_m \neq b_{m+i} \quad \text{for} \quad i = 0, 1, 2, \cdots;$$

hence, by (2) and induction,

$$(3) \qquad a_m = a_{m+\iota} \quad \text{for} \quad i = 0, 1, 2, \cdots.$$

From (1) and (3) we easily conclude with the help of 15.2(ii),(iii) that

$$a_n = a_m + \sum_{\iota<\infty} b_{n+i} \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

Thus, the element $c = a_m$ satisfies the conclusion of 5.1.V. Therefore, by 15.9, $\mathfrak{A}$ is a G.C.A.

This conclusion holds *a fortiori* if $\mathfrak{A}$ is simply ordered and satisfies (i). In this case, however, we infer by 15.2(iii) that $\mathfrak{A}$ is not only a G.C.A. but also a C.A.; and the proof is complete.

From 15.12 we see that the bound $\bigcap_{i<\infty} a_i$ does not always exist in a C.A. $\mathfrak{A}$, even if all the bounds $\bigcap_{i<n} a_i$ with $n < \infty$ exist. We see also that a lattice $\mathfrak{A}$ which is a C.A. or a G.C.A. is not necessarily countably complete, even in the wider sense.

An especially important class of lattices is constituted by BOOLEAN ALGEBRAS:

DEFINITION 15.13. $\mathfrak{A} = \langle A, + \rangle$ *is called a* BOOLEAN ALGEBRA *if it is a finitely distributive lattice satisfying the following condition:*

(*i*) *for any* $a, b \in A$ *with* $a \leqq b$ *there is an element* $c \in A$ *such that* $a + c = b$ *and* $a \cap c = 0$.

Here we do not require a Boolean algebra to have a largest element, i.e., the unit element 1; thus, Boolean algebras in the sense of 15.14 are what are sometimes called GENERALIZED BOOLEAN ALGEBRAS.[10]

$\mathfrak{A} = \langle A, + \rangle$ being a Boolean algebra, it is easily seen from 15.13(i) that the $\leqq$ relations in $\mathfrak{A}$ and in the correlated disjunctive algebra $\dot{\mathfrak{A}} = \langle A, \dot{+} \rangle$ (cf. 8.16) coincide. Hence, the relation $\leqq$ can be defined in terms of the disjunctive addition $\dot{+}$. This implies further that the fundamental operation $+$ of $\mathfrak{A}$ can itself be defined in terms of $\dot{+}$. Consequently, we can treat Boolean algebras as systems in which $\dot{+}$ plays the role of the (unique) fundamental operation.[11] This conception of Boolean algebras proves useful in certain situations; this will be seen, e.g., in the appendix to our work. We want to characterize here Boolean algebras thus conceived by means of a separate postulate system. We define (by using '$+$' instead of '$\dot{+}$'):

DEFINITION 15.14. *An algebra* $\mathfrak{A} = \langle A, + \rangle$ *is called a* DISJUNCTIVE BOOLEAN ALGEBRA *if it satisfies the commutative and associative laws* 11.26.I, II, *has a zero element, is multiple-free, has the refinement property, and satisfies the following condition:*

(i) *if* $a, b \in A$, *then there are elements* $a', b', c \in A$ *such that* $c = a + a' = b + b'$ (*i.e., there is an element* $c \in A$ *such that* $a \leqq c$ *and* $b \leqq c$).

[10] This notion was introduced in Stone [1].

[11] It may be interesting to mention that G. Boole, in his study of new algebras which were later called after his name, used exclusively the disjunctive addition; cf. Boole [1], pp. 35 f. and 61 ff.

We could also characterize disjunctive Boolean algebras as multiple-free R.A.'s which satisfy 15.14(i).

THEOREM 15.15. *If* $\mathfrak{A} = \langle A, + \rangle$ *is a Boolean algebra (in the sense of* 15.13), *then* $\dot{\mathfrak{A}} = \langle A, \dot{+} \rangle$ *is a disjunctive Boolean algebra (in the sense of* 15.14); *the zero elements and the* $\leqq$ *relations in* $\mathfrak{A}$ *and* $\dot{\mathfrak{A}}$ *coincide.*

The proof is based upon familiar elementary properties of Boolean algebras (in the usual sense of this term), and need not be given here.

THEOREM 15.16. *If* $\mathfrak{A} = \langle A, + \rangle$ *is a disjunctive Boolean algebra, then* $\breve{\mathfrak{A}} = \langle A, \cup \rangle$ *is a Boolean algebra; the zero elements and the* $\leqq$ *relations in* $\mathfrak{A}$ *and* $\breve{\mathfrak{A}}$ *coincide.*

PROOF: By invariably applying 15.14, we establish, step by step, the following properties of arbitrary elements in A.

(1) $\qquad a \leqq a;\quad$ if $a \leqq b$ and $b \leqq c$, then $a \leqq c$.

This follows from the fact that $\mathfrak{A}$ has a zero element and satisfies the associative law 5.9(i).

(2) $\qquad$ If $a + b = a$ or $a + b = 0$, then $b = 0$.

This can easily be derived from 5.8, 5.9(i), and from the fact that $\mathfrak{A}$ is multiple-free.

(3) $\qquad$ If $a \leqq b$ and $b \leqq a$, then $a = b$.

For the formulas

$$a + c = b \quad \text{and} \quad b + d = a$$

imply by 5.8, 5.9(i), and (2)

$$c = d = 0,$$

and hence $a = b$. By (1) and (3), $\mathfrak{A}$ is partially ordered.

(4) $\qquad$ If $a + b \;\varepsilon\; A$, then $a \cap b = 0$.

This holds for the same reasons as (2).

(5) $\qquad$ If $a \cap b = 0$, then $a + b \;\varepsilon\; A$ and $a + b = a \cup b$.

In fact, if

$$a \leqq x \quad \text{and} \quad b \leqq x,$$

i.e.,

$$a + a' = b + b' = x \quad \text{for some } a' \text{ and } b',$$

then by 2.3 (which holds in $\mathfrak{A}$ in view of 13.18) we have

$$a = c_1 + c_2, \quad a' = c_3 + c_4, \quad b = c_1 + c_3, \quad \text{and} \quad b' = c_2 + c_4.$$

Since $a \cap b = 0$, we obtain

$$c_1 = 0, \quad a = c_2, \quad b = c_3, \quad x = a + (b + c_4);$$

and therefore, by 5.9(i), $a + b$ is in A and

$$a + b \leqq x.$$

On the other hand, by 15.14(i), there is an element $x \varepsilon A$ such that

$$a \leqq x \quad \text{and} \quad b \leqq x.$$

Hence, by the argument just carried through, $a + b$ is in A; and we obviously have

$$a \leqq a + b \quad \text{and} \quad b \leqq a + b.$$

Consequently, by 3.2 and (3),

$$a + b = a \cup b.$$

(6) If $b + c \varepsilon A$ and $a \leqq b + c$, then $a = a_1 + a_2$, $a_1 \leqq b$, and $a_2 \leqq c$ for some $a_1, a_2 \varepsilon A$.

(7) If $b + c \varepsilon A$, $a \leqq b + c$, and $a \cap c$ exists, then $a \leqq b + (a \cap c)$.

(6) and (7) can be derived from 2.3 in the same way as 2.4 and 3.3.

(8) $a \cap b$ always exists.

In fact, we have by 5.14(i), for some $b', c \varepsilon A$,

$$a \leqq b + b' = c.$$

Hence, by (6) and (4),

$$a = a_1 + a_2, \quad a_1 \leqq a, \quad a_1 \leqq b, \quad a_2 \leqq b', \quad b \cap b' = 0.$$

Now if

$$x \leqq a \quad \text{and} \quad x \leqq b,$$

then

$$x \leqq a_1 + a_2 \quad \text{and} \quad x \cap a_2 = 0,$$

and therefore, by (7),

$$x \leqq a_1 .$$

Consequently,

$$a_1 = a \cap b.$$

(9) $a \cup b$ always exists.

This we derive from (8) with the help of (7), by arguing in practically the same way as in the proof of 3.4; we make, however, an essential use of 15.14(i).

(10) If $a + b = a \cup c \,\varepsilon\, A$, then $b \leqq c$.

For the hypothesis of (10) implies by (7) and (8)

$$c \leqq a + (b \cap c) \quad \text{and} \quad a \leqq a + (b \cap c);$$

hence

$$a + b = a \cup c \leqq a + (b \cap c) \quad \text{and} \quad b \leqq a + (b \cap c);$$

and therefore, by (4) and (7),

$$b \leqq (a \cap b) + (b \cap c) = b \cap c \quad \text{and} \quad b \leqq c.$$

(11) $a \cap (b \cup c) = (a \cap b) \cup (a \cap c).$

In fact, we have by (9) and (10), for some $c' \,\varepsilon\, A$,

$$b \cup c = b + c' \quad \text{and} \quad c' \leqq c.$$

Hence, by applying (7) twice, with 'a' replaced by '$a \cap (b \cup c)$', we obtain in view of (8):

$$a \cap (b \cup c) \leqq (a \cap b) + (a \cap c');$$

therefore, by (4) and (5),

$$a \cap (b \cup c) \leqq (a \cap b) \cup (a \cap c');$$

and further, with the help of (8) and (9),

$$a \cap (b \cup c) \leqq (a \cap b) \cup (a \cap c).$$

The inequality in the opposite direction being obvious, we arrive at (11).

We turn now to the algebra $\breve{\mathfrak{A}} = \langle A, \cup \rangle$. From (1), (3), (8), and (9) we easily see that the $\leqq$ relations and the zero elements in $\mathfrak{A}$ and $\breve{\mathfrak{A}}$ coincide, and that $\breve{\mathfrak{A}}$ is a lattice in the sense of 15.1 (we apply 8.1 and 13.22 here). Hence, by 15.5 and (11), $\breve{\mathfrak{A}}$ is a distributive lattice; and, by (4), $\breve{\mathfrak{A}}$ satisfies 15.13(i). Thus, $\breve{\mathfrak{A}}$ is a Boolean algebra; and this is what we wanted to prove.

THEOREM 15.17. *For every algebra* $\mathfrak{A} = \langle A, + \rangle$ *the following three conditions are equivalent:*

(i) $\mathfrak{A}$ *is a Boolean algebra;*

(ii) $\dot{\mathfrak{A}}$ *is a disjunctive Boolean algebra and* $\mathfrak{A} = \breve{\dot{\mathfrak{A}}}$;

(iii) *there is a disjunctive Boolean algebra* $\mathfrak{B}$ *such that* $\mathfrak{A} = \breve{\mathfrak{B}}$.

PROOF: From 15.2(ii), 15.13, and 15.15 we easily see that (i) implies (ii); (ii) obviously implies (iii); and (iii) implies (i) by 15.16.

THEOREM 15.18. *For every algebra* $\mathfrak{A} = \langle A, + \rangle$ *the following three conditions are equivalent:*

(i) $\mathfrak{A}$ *is a disjunctive Boolean algebra;*

(ii) $\breve{\mathfrak{A}}$ *is a Boolean algebra and* $\mathfrak{A} = \dot{\breve{\mathfrak{A}}}$;

(iii) *there is a Boolean algebra* $\mathfrak{B}$ *such that* $\mathfrak{A} = \dot{\mathfrak{B}}$.

PROOF: To derive (ii) from (i), we apply 15.16, and we also use lemmas (4) and (5) stated in the proof of 15.16 (which show that the operations $+$ and $\cup$ in a disjunctive Boolean algebra coincide whenever the involved elements are disjoint). Furthermore, (ii) obviously implies (iii); and (iii) implies (i) by 15.15.

We can repeat here the remarks made in §8 in connection with 8.19 and 8.20. Theorems 15.17 and 15.18 show that ordinary Boolean algebras and disjunctive Boolean algebras are essentially the same algebras, although characterized in terms of different fundamental operations.

It is easily seen that the cancellation law for sums holds in disjunctive Boolean algebras (under the assumption that the sums involved exist). Thus, every disjunctive Boolean algebra $\mathfrak{A}$ is what we have called a generalized semigroup. Since, in addition, $\mathfrak{A}$ is partially ordered and has the refinement property, it can be imbedded in a partially ordered semigroup $\mathfrak{A}'$ with the refinement property, which is a finite closure of $\mathfrak{A}$; and such a semigroup $\mathfrak{A}'$ is determined by $\mathfrak{A}$ up to isomorphism (cf. the remarks at the end

of §13). Moreover, $\mathfrak{A}$ proves to be the only disjunctive Boolean algebra of which $\mathfrak{A}'$ is a finite closure; $\mathfrak{A}$ can be characterized as the subalgebra of $\mathfrak{A}'$ constituted by the set of all multiple-free elements. Consequently, two disjunctive Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic if, and only if, the correlated semigroups $\mathfrak{A}'$ and $\mathfrak{B}'$ are isomorphic.

We thus have a perfect correspondence (up to isomorphism) between disjunctive Boolean algebras and a certain class of semigroups which are finite closures of these algebras. The semigroups involved can be referred to as BOOLEAN SEMIGROUPS. They can be 'intrinsically' characterized as (commutative) semigroups with the refinement property in which every element is a sum of finitely many multiple-free elements and in which any two multiple-free elements a and b have a multiple-free upper bound (i.e., a multiple-free element c with $a \leqq c$ and $b \leqq c$). Hence some further properties of Boolean semigroups can be derived; e.g., they are finitely complete, and consequently they satisfy the cancellation laws for multiples 2.33 and 2.34, as well as Euclid's theorem 2.37 (cf. the remarks which follow 13.28).

The following examples will serve to illustrate the foregoing considerations. The additive semigroup $\mathfrak{J}$ of finite non-negative integers (cf. 14.1) is clearly a Boolean semigroup; its only multiple-free elements are 0 and 1; and $\mathfrak{J}$ is a finite closure of the disjunctive Boolean algebra constituted by these two numbers under ordinary addition. On the other hand, the multiplicative semigroup $\mathfrak{M}$ of finite positive integers is also a Boolean semigroup; its multiple-free elements are all square-free integers; and $\mathfrak{M}$ is a finite closure of the disjunctive Boolean algebra formed by all square-free integers under ordinary multiplication.

We are presented here with a new method of including the theory of Boolean algebras in the main current of modern algebraic investigations—a method which is essentially different from the one which can be found in the literature.[12] This method permits us to obtain some new results for Boolean algebras. We see, for instance, that the theorem on the extension of partial order to simple order applies to all disunctive Boolean algebras (cf. again the remarks which follow 13.28).

[12] We, of course, have in mind the well-known result of M. H. Stone which subsumes the theory of Boolean algebras under the general theory of commutative rings; cf. Stone [2], pp. 43 ff.

Returning to our main topic after this digression, we first state without proof the following two familiar theorems:

THEOREM 15.19. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be a Boolean algebra (or a disjunctive Boolean algebra).*

(i) *For* $\mathfrak{A}$ *to be countably complete it is necessary and sufficient that it satisfy* 15.12(i).

(ii) *For* $\mathfrak{A}$ *to be countably complete in the wider sense it is necessary and sufficient that it satisfy* 15.12(ii).

THEOREM 15.20. *Every Boolean algebra (and every disjunctive Boolean algebra) is completely distributive, and hence also countably distributive.*

For a correct interpretation of this theorem compare the remarks which follow 15.4.[13]

Connections between C.A.'s (or G.C.A.'s) and Boolean algebras will be established in Theorems 15.21–15.24.

THEOREM 15.21. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be a G.C.A.*

(i) *For* $\mathfrak{A}$ *to be a Boolean algebra it is necessary and sufficient that it be finitely closed and satisfy* 15.13(i).

(ii) *For* $\mathfrak{A}$ *to be a countably complete Boolean algebra it is necessary and sufficient that it be a C.A. and again satisfy* 15.13(i).

PROOF: The necessity of the conditions follows from 15.6 and 15.13. If the conditions in the first part of the theorem are satisfied, then, by 8.19, $\mathfrak{A}$ is idem-multiple. Furthermore, for any $a, b \,\varepsilon\, A$ there is a $c \,\varepsilon\, A$ such that

$$b + a = b + c \quad \text{and} \quad b \cap c = 0.$$

Consequently, $a \cap (b + c)$ exists, and therefore $a \cap b$ exists by 3.13. Thus, by 15.6(i), $\mathfrak{A}$ is lattice; and finally, by 3.30, 3.32, 15.4 (ii), and 15.13, $\mathfrak{A}$ is a Boolean algebra. The sufficiency of the conditions in the second part of the theorem follows easily by 15.2(iii), 15.13, and 15.19.

[13] Theorem 15.20 for complete Boolean algebras was stated in Tarski [3], p. 510 (footnote); in a more general form, but which still is weaker than that given in the text, it was established in von Neumann [1], Part III, p. 7. Von Neumann's argument can be used, with small changes, to obtain the proof of Theorem 15.20 in its form actually given.

THEOREM 15.22. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be a G.C.A.*

(i) *For* $\mathfrak{A}$ *to be a disjunctive Boolean algebra it is necessary and sufficient that* $\mathfrak{A}$ *be multiple-free and satisfy* 15.14(i).

(ii) *For* $\mathfrak{A}$ *to be a countably complete disjunctive Boolean algebra it is necessary and sufficient that* $\mathfrak{A}$ *be multiple-free and that, for any* a_0, $a_1, \cdots, a_i, \cdots \varepsilon A$, *there exist an element* $b \varepsilon A$ *such that* $a_i \leqq b$ *for* $i = 0, 1, 2, \cdots$.

PROOF: The first part of the theorem follows obviously from 15.14 (with the help of 2.3 and 13.18). The necessity of the conditions in the second part of the theorem results from 13.22 and 15.14; if these conditions are satisfied, then $\mathfrak{A}$ is a Boolean algebra by the first part of the theorem, and is countably complete by 4.36, 8.13(i), (ii), and 15.19.

THEOREM 15.23. *For a Boolean algebra* $\mathfrak{A} = \langle A, + \rangle$ *to be a C.A. (or a G.C.A.) it is necessary and sufficient that* $\mathfrak{A}$ *be countably complete (or countably complete in the wider sense).*

PROOF: by 15.10, 15.12, 15.13, 15.19, and 15.20.

THEOREM 15.24. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be a disjunctive Boolean algebra. Then* $\mathfrak{A}$ *is never a C.A. or a finitely closed G.C.A., unless A consists of one element only; and for* $\mathfrak{A}$ *to be a G.C.A. it is necessary and sufficient that* $\mathfrak{A}$ *be countably complete in the wider sense.*

PROOF: The first part of the theorem follows from 8.12 and 15.14. To obtain the second part, we notice that, by 13.23 and 15.16, $\mathfrak{A}$ is countably complete in the wider sense if, and only if, the same holds for $\breve{\mathfrak{A}}$; and then we apply 8.14, 8.17, 15.14, 15.18, and 15.23.

From 15.22 and 15.24 we see that the class of multiple-free G.C.A.'s is not much wider than that of disjunctive Boolean algebras which are countably complete (in the wider sense). We should not lose very much if we stated all our results concerning multiple-free G.C.A.'s, or disjunctive G.C.A.'s $\dot{\mathfrak{A}}$, (e.g., various results of §11) as theorems on countably complete Boolean algebras.

Given a G.C.A. $\mathfrak{A}$, we can ask the question under what conditions, not $\mathfrak{A}$ itself, but $\breve{\mathfrak{A}}$ is a lattice or a Boolean algebra, or $\dot{\mathfrak{A}}$ a disjunctive Boolean algebra. The answer is not difficult, but will not be stated here.

Turning to more special results, we notice that the fundamental theorem on ideal algebra, 10.3, and the related theorems 10.4–10.7 and 10.9 can now be given the following form:

THEOREM 15.25. *$\mathfrak{A} = \langle A, +\rangle$ being an arbitrary G.C.A., the algebra $\mathfrak{J}(\mathfrak{A}) = \langle \mathsf{I}, +\rangle$ (where I is the family of all ideals in $\mathfrak{A}$, and $+$ is the ideal addition defined in* 10.1) *is a complete lattice; this lattice is countably distributive, and satisfies the unrestricted distributive law* 10.9. *In the lattice $\mathfrak{J}(\mathfrak{A})$ the zero element is the ideal $I(0)$, the relation $\leq$ coincides with set-theoretical inclusion, and greatest lower bounds coincide with set-theoretical intersections.*

PROOF: by 10.3–10.7 and 10.9, and with the help of 13.22, 15.4(ii), and 15.6.

COROLLARY 15.26. *The conclusions of Theorem* 15.25 *apply to every lattice $\mathfrak{A} = \langle A, +\rangle$ which is countably complete (if only in the wider sense) and countably distributive, and in particular to every countably complete Boolean algebra (as well as to every countably complete disjunctive Boolean algebra).*

PROOF: by 15.10 and 15.23–15.25.

In connection with this corollary it should be remembered that ideals in the sense of 9.1 are called in lattice theory COUNTABLY CLOSED IDEALS. Corollary 15.26 can easily be established directly, without the help of 15.25; the only detail in this corollary which presents some interest is the fact that the lattice of (countably closed) ideals is countably distributive.[14]

Among other results which can be obtained as applications of the general theory of C.A.'s and G.C.A.'s, the following corollaries from 12.17 and 12.21 are perhaps the most important from the point of view of lattice theory:

THEOREM 15.27. *If T is the class of isomorphism types of all countably complete and countably distributive lattices, or of all countably complete Boolean algebras, then the algebra $\langle \mathrm{T}, \times\rangle$ (as well as $\langle \mathrm{T}, \times, \prod\rangle$) is a C.A.*

PROOF: We easily see that Theorem 12.9 applies to types of lattices and Boolean algebras involved in 15.27, and that the type 1 defined in 12.6 is in T. Hence, and by 9.1, 15.10, and 15.23, the set T is an ideal in the algebra $\mathfrak{T}$ of 12.17. The conclusion follows by 6.13, 9.18(ii), and 13.1.

[14] A result analogous to 15.26, but which applies to all ideals of an arbitrary distributive lattice (and therefore does not involve the countable distributivity of the ideal algebra), can be found in Stone [3], pp. 3 ff.

THEOREM 15.28. *If* T *is the class of isomorphism types of countably distributive lattices, or Boolean algebras, which are countably complete in the wider sense, then* $\langle \mathrm{T}, \times \rangle$ *is a finitely closed R.A.*

PROOF: analogous to that of 13.30, with 13.27 and 13.28 replaced by 15.10 and 15.23.

Theorems 15.27 and 15.28 apply also to types of disjunctive Boolean algebras. In fact, it is easily seen that the algebras of isomorphism types of Boolean algebras and of disjunctive Boolean algebras are isomorphic (under cardinal multiplication).

The arithmetic of lattices is much simpler than that of C.A.'s. All the arithmetical results of Part I become trivial when applied to C.A.'s and G.C.A.'s which are lattices. The situation changes, however, when we apply these results to algebras which are constructed from lattices by means of methods discussed in Part II, but which are not lattices themselves. For instance, the fact that all theorems of Part I hold for cardinal products of isomorphism types of countably complete Boolean algebras—or, more generally, countably complete and countably distributive lattices—seems to be of real interest.

§ 16. ALGEBRAS OF SETS

Our discussion in the remaining part of this work will be somewhat less systematic than it has been so far. In particular, we shall use various notions taken in the main from the general theory of sets, without stating their definitions in a formal way.

We shall be concerned in the present section with algebras constituted by FAMILIES OF SETS under the operation of SET-THEORETICAL ADDITION—the formation of UNIONS.[15] In discussing these algebras and certain derived algebraic systems we find it more advantageous to return to our original conception of C.A.'s and G.C.A.'s, i.e., to regard C.A.'s and G.C.A.'s as systems with two fundamental operations, $+$ and $\sum$.

The union of two sets A and B will be denoted by '$A + B$'; more generally, the union of all sets A_i correlated with elements i of a given set I will be denoted by '$\sum_{i \in I} A_i$'.

We shall not hesitate here to speak of the class of all sets (cf. the remarks which follow Definition 12.1). Hence we state the following:

THEOREM 16.1. *Let* S *be the class of all sets.*

(i) *The algebra* $\langle \mathsf{S}, + \rangle$ *is a Boolean algebra which is countably complete (in the strict sense, and simply complete in the wider sense), and the algebra* $\mathfrak{S} = \langle \mathsf{S}, +, \sum \rangle$ *is a C.A.*

(ii) *In these algebras, the zero element is the empty set, the relation* $\leqq$ *coincides with that of inclusion, least upper bounds coincide with unions, and greatest lower bounds coincide with intersections.*

PROOF: The theorem is partly well known, and partly follows from 15.23 (with the help of 13.1–13.3).

The answer to the question whether the algebra $\langle \mathsf{S}, + \rangle$ is complete in the strict sense (i.e., whether $\mathfrak{S}$ contains a largest set) depends on the set-theoretical foundations which are accepted as a base of the discussion.

In view of 16.1, we can use the symbols '0', '$\leqq$', and '$\cap$' (or '$\bigcap$') to denote the empty set, the relation of INCLUSION, and the operation of SET-THEORETICAL MULTIPLICATION—the formation of INTER-

[15] For various notions involved in this section consult Hausdorff [1] (set-fields, set-rings, the geometric problem of measure, Lebesgue measure) and Kuratowski [2] (metric spaces, open and Borelian sets).

SECTIONS; and the symbols '$+$' and 'U', or '$\sum$' and 'U', can be used interchangeably to denote the operation of set-theoretical addition. The operation of SET-THEORETICAL SUBTRACTION will be denoted by '$-$'; thus, A and B being two arbitrary sets, $A - B$ is the set of all elements which are in A but not in B.

Among cardinal subalgebras of the C.A. $\mathfrak{S}$ of 16.1 those formed by set-fields deserve special attention. A non-empty family F of sets is called a SET-FIELD if, together with any two sets X and Y, F contains also their sum $X + Y$ and difference $X - Y$; if in addition F contains the sums of all infinite sequences, or of arbitrary collections, of its sets, it is referred to as a COUNTABLY COMPLETE SET-FIELD (a σ-FIELD), or a COMPLETE SET-FIELD. The family of all subsets of a given set A is clearly a complete set-field.

THEOREM 16.2. *The family* F *being a countably complete set-field, the algebra* $\langle \mathsf{F}, + \rangle$ *is a countably complete Boolean algebra; and the algebra* $\langle \mathsf{F}, +, \sum \rangle$ *is a C.A., and hence a cardinal subalgebra of the algebra* $\mathfrak{S}$ *of* 16.1. *Part* (ii) *of* 16.1 *applies to these algebras.*

PROOF: The theorem is partly well known, and partly follows from 15.23 and 16.1 (with the help of 6.13 and 13.1–13.3).

The C.A. $\mathfrak{S}$ contains also many other cardinal subalgebras. We want to mention here subalgebras constituted by set-rings. A non-empty family F of sets is called a SET-RING if, together with any two sets, it contains also their sum and product; if the same applies to sums and products of infinite sequences of sets, F is called a COUNTABLY COMPLETE SET-RING. Every set-field is a set-ring, and every countably complete set-field is a countably complete set-ring. F being a countably complete set-ring which has a smallest set Z, the algebra $\langle \mathsf{F}, + \rangle$ is clearly a countably complete lattice, and the algebra $\langle \mathsf{F}, +, \sum \rangle$ is a C.A. To support some remarks made in §1 in connection with Definition 1.1, let us consider still algebras $\langle \mathsf{F}, +, \sum \rangle$ formed by those set-rings F which are closed under countable addition, but not under countable multiplication. These algebras are not always C.A.'s, though they show much similarity to C.A.'s. They satisfy Postulates 1.1.I–VI and also various consequences of 1.1.VII, e.g., Theorem 2.21; almost all theorems of Part I obviously hold in these algebras; nevertheless, Postulate 1.1.VII may fail. To obtain an example, take for F the family of all open sets of a Euclidean space. We can easily construct two infinite sequences

of open sets A_n and B_n which satisfy the hypothesis of 1.1.VII as well as the following condition: the intersection of all sets A_n is not an open set and is disjoint with each of the sets B_n. A simple argument shows that in this case no open set C can satisfy the conclusion of 1.1.VII.

The problem of characterizing algebraically those C.A.'s $\mathfrak{A}$ which are isomorphic with the algebras discussed in 16.1 and 16.2 seems to be difficult—especially if we wish this characterization to be given in terms of arithmetical properties of elements of $\mathfrak{A}$. It might seem plausible, for instance, that the following conditions are necessary and sufficient for an algebra $\mathfrak{A} = \langle A, + \rangle$ to be isomorphic with an algebra $\langle \mathsf{F}, + \rangle$ of 16.2: $\mathfrak{A}$ is a countably complete Boolean algebra and it satisfies, for every double sequence of elements $a_{i,j} \varepsilon A$, the general law of countable distributivity, i.e.,

$$\bigcap_{i<\infty} \bigcup_{j<\infty} a_{i,j} = \bigcup_{k \varepsilon S} \bigcap_{i<\infty} a_{i,k_i},$$

where S is the set of all infinite sequences of finite integers. It can be shown, however, that these conditions are not sufficient. On the other hand, such algebras $\mathfrak{A}$ can be characterized as countably complete Boolean algebras in which every element $a \neq 0$ belongs to a (COUNTABLY CLOSED) PRIME IDEAL, i.e., to an ideal $B \neq A$ which is not included in any ideal different from A and B.

However, we meet with no difficulty if we try to characterize algebraically all those C.A.'s and G.C.A.'s $\mathfrak{A}$ which are isomorphic with arbitrary subalgebras of $\mathfrak{S}$. In fact, we have

THEOREM 16.3. *For every G.C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$ *the following three conditions are equivalent:*

(i) $\mathfrak{A}$ *is idem-multiple;*

(ii) $\mathfrak{A}$ *is isomorphic with a subalgebra of the C.A.* $\mathfrak{S}$ *of* 16.1; *i.e., with an algebra* $\langle \mathsf{F}, +, \sum \rangle$ *constituted by a family of sets under the operations of set-theoretical addition;*

(iii) $\mathfrak{A}$ *is isomorphic with a subalgebra of a C.A.* $\mathfrak{B} = \langle B, +, \sum \rangle$ *where* $\langle B, + \rangle$ *is a lattice.*

PROOF: To show that (i) implies (ii), we assume $\mathfrak{A}$ to be idem-multiple, and we correlate, with every a in A, the set $G(a)$ of all elements $x \varepsilon A$ for which $a \leqq x$ does not hold. F being the family of all sets thus obtained, it is easily seen from 6.1, 8.1, and 8.2 that G maps $\mathfrak{A}$ isomorphically onto $\langle \mathsf{F}, +, \sum \rangle$, i.e., by 6.13, onto a

subalgebra of $\mathfrak{S}$. The implications between (ii) and (iii), and between (iii) and (i), are obvious; we apply 8.1, 15.1, 15.13, and 16.1. The proof is thus complete.

From C.A.'s $\langle \mathsf{F}, +, \sum \rangle$ of 16.2 we can obtain other C.A.'s by applying, e.g., the method of 6.10, i.e., by constructing coset algebras generated by additive and refining equivalence relations. The coset algebras thus constructed present no special interest; as is easily seen, they are all idem-multiple, and their arithmetic is trivial. We obtain, on the other hand, really interesting examples of G.C.A.'s if we replace $\langle \mathsf{F}, +, \sum \rangle$ in this construction by the correlated disjunctive algebras $\langle \mathsf{F}, \dot{+}, \dot{\sum} \rangle$. The following two theorems, 16.4 and 16.5, which apply to these disjunctive algebras, are immediate or almost immediate corollaries of 6.10 and 7.13; nevertheless, they seem worth while formulating explicitly here, for they summarize probably the most interesting implications of the theory of C.A.'s in the general theory of sets. For typographical reasons we use in these theorems the symbol '$\equiv$', instead of the letter 'R', to denote a variable equivalence relation between sets.

THEOREM 16.4. *Let* F *be a countably complete set-field. Let* $\equiv$ *be an equivalence relation in* F *which satisfies, for all sets* $A, A_0, A_1, \cdots, B, B_0, B_1, \cdots$ *of* F, *the following conditions:*

(i) *if* $A = \dot{\sum}_{i<\infty} A_i$, $B = \dot{\sum}_{i<\infty} B_i$, *and* $A_i \equiv B_i$ *for* $i = 0, 1, 2, \cdots$, *then* $A \equiv B$;

(ii) *if* $A = A_1 \dot{+} A_2$ *and* $A \equiv B$, *then there are sets* B_1 *and* B_2 *in* F *such that* $B = B_1 \dot{+} B_2$, $A_1 \equiv B_1$, *and* $A_2 \equiv B_2$; *in other words, let* $\equiv$ *be an infinitely additive and finitely refining equivalence relation in the algebra* $\mathfrak{F} = \langle \mathsf{F}, \dot{+}, \dot{\sum} \rangle$. *Under these assumptions, the algebra* $\mathfrak{F}/\equiv$ *is a G.C.A.*

PROOF: by 6.4, 6.7, 6.10, 8.17, and 16.2.

It will be seen from our further discussion that the variety of G.C.A.'s which can be represented as coset algebras $\langle \mathsf{F}, \dot{+}, \dot{\sum} \rangle/\equiv$ of 16.4 is very large, and that their algebraic structure may differ very considerably from that of the original algebra $\langle \mathsf{F}, +, \sum \rangle$ or $\langle \mathsf{F}, \dot{+}, \dot{\sum} \rangle$. A rather general REPRESENTATION THEOREM may be expected in this domain. As a matter of fact, it can easily be shown that every C.A. $\mathfrak{A}$ is isomorphic with a coset algebra $\langle \mathsf{F}, \dot{+}, \dot{\sum} \rangle/\equiv$ where F is a countably complete set-field, and where

$\equiv$ is an equivalence relation in F which satisfies 16.4(i), but not necessarily 16.4(ii), i.e., which is infinitely additive, but not refining. The result is rather trivial, and can be extended to arbitrary algebras $\mathfrak{A} = \langle A, +, \sum \rangle$, which satisfy the closure postulates and the general commutative and associative laws, and which have a zero element 0 with the properties stated in 1.6 and 1.7. (Thus, Postulates 1.1.VI and 1.1.VII, which are the most characteristic for C.A.'s, may fail in these algebras.) In fact, $\mathfrak{A} = \langle A, +, \sum \rangle$ being an algebra of this kind, let K be the set of all ordered couples $\langle a, k \rangle$ with a in A and $k < \infty$, and let F be the family of all at most denumerable subsets of K. F is obviously a countably complete set-field. B and C being any two sets in F, we arrange their elements in finite or infinite sequences without repetitions: $\langle b_0, k_0 \rangle, \langle b_1, k_1 \rangle, \cdots, \langle b_i, k_i \rangle, \cdots$ with $i < m \leqq \infty$, and $\langle c_0, l_0 \rangle, \langle c_1, l_1 \rangle, \cdots, \langle c_j, l_j \rangle, \cdots$ with $j < n \leqq \infty$; and we define $\equiv$ as the relation which holds between B and C if, and only if,

$$\sum_{i<m} b_i = \sum_{j<n} c_j.$$

The proof that $\mathfrak{A}$ is isomorphic with $\langle \mathsf{F}, \dot{+}, \dot{\sum} \rangle / \equiv$ is obvious. An analogous result applies to arbitrary groupoids $\mathfrak{A} = \langle A, + \rangle$; in this case, however, the set-field F is not countably additive, and the equivalence relation $\equiv$ is finitely, but not infinitely, additive.

THEOREM 16.5. *Under the hypothesis of* 16.4, *we have for any sets* $A, A', A_0, A_1, \cdots, B, \cdots, C, \cdots$ *in* F:

(i) *if* $A \leqq B$, $A' \leqq B'$, $A \equiv B'$, *and* $B \equiv A'$, *then* $A \equiv A'$ *and* $B \equiv B'$;

(ii) *if* $A \leqq B \leqq C$ *and* $A \equiv C$, *then* $A \equiv B \equiv C$;

(iii) *if* $n \leqq \infty$, *then*

$$\sum_{i<n} A_i + B \equiv B$$

implies that $A_i + B \equiv B$ *for every* $i < n$, *and conversely;*

(iv) *if* $A_0 \dot{+} C_0 \dot{+} C_2 = A$, $B_0 \dot{+} C_1 \dot{+} C_3 = B$, $A \equiv B$, *and* $C_0 \equiv C_i$ *for* $i = 1, 2, 3$, *then* $A_0 + C_0 \equiv B_0 + C_1$;

(v) *if* $0 < m < \infty$, $A_0 \equiv A_i$ *and* $B_0 \equiv B_i$ *for every* $i < m$, *and* $A \equiv B$ *where*

$$A = \dot{\sum_{i<m}} A_i \quad \textit{and} \quad B = \dot{\sum_{i<m}} B_i,$$

then $A_0 \equiv B_0$;

(vi) *if* $A \leqq B \leqq C$, $A' \leqq C'$, $A \equiv A'$, *and* $C \equiv C'$, *then there is a set* B' *in* F *such that* $A' \leqq B' \leqq C'$ *and* $B \equiv B'$;

(vii) *if there is a set* A' *in* F *such that* $A \equiv A' \leqq B$, *then there is also a set* B' *in* F *such that* $A \leqq B' \equiv B$, *and conversely;*

(viii) *if, for every* $n < \infty$, *there is a set* D_n *in* F *such that*

$$\sum_{i<n} A_i \equiv D_n \leqq B,$$

then there is a set D *in* F *such that*

$$\sum_{i<\infty} A_i \equiv D \leqq B.$$

Proof: In view of 8.17 and 16.2, conclusions (iv) and (v) follow directly from the corresponding conclusions in 7.13; and since, by 8.16, 15.6, and 16.2, the relations $\leqq$ in $\langle \mathsf{F}, +, \sum \rangle$ and $\dot{\leqq}$ in $\langle \mathsf{F}, \dot{+}, \dot{\sum} \rangle$ coincide, the same applies to conclusions (i), (ii), and (vi).

To obtain (iii), we reason as follows. Let

(1) $$\sum_{i<n} A_i + B \equiv B.$$

We obviously have

(2) $$B \leqq A_i + B \leqq \sum_{i<n} A_i + B \quad \text{for every} \quad i < n.$$

By applying conclusion (ii), we obtain from (1) and (2)

(3) $$A_i + B \equiv B \quad \text{for every} \quad i < n.$$

If, conversely, (3) holds, we put

(4) $$A'_k = A_k - (\sum_{i<k} A_i + B) \quad \text{for every} \quad k < n.$$

Since

$$B \leqq A'_i \dot{+} B \leqq A_i + B,$$

we obtain by (ii) and (3)

$$A'_i \dot{+} B \equiv B \quad \text{for every} \quad i < n;$$

and hence, by 7.13(iii), 8.17, and 16.2,

(5) $$\dot{\sum_{i<n}} A'_i \dot{+} B \equiv B.$$

(4) and (5) at once imply (1); thus, (iii) holds. In a similar way we derive (viii) from 7.13(vii). Finally, to derive (vii), assume that

$$A \equiv A' \leqq B.$$

Thus,

$$A' \leqq B \leqq A + B, \quad A \leqq A + B, \quad \text{and} \quad A' \equiv A;$$

and, by applying (vi), we conclude that there is a set B' in F with

$$A \leqq B' \equiv B.$$

The implication in the opposite direction follows immediately from 6.8. This completes the proof.

Notice that a conclusion analogous to 16.5(vii) is lacking in 7.13. The reason is that such a conclusion fails in arbitrary G.C.A.'s. It is easily seen that this conclusion holds in C.A.'s and finitely closed G.C.A.'s; our argument in the proof of 16.5 shows, however, that it also holds in a wider class of G.C.A.'s—in fact, in those G.C.A.'s in which any two elements have a common upper bound.

We arrive at an important class of equivalence relations to which Theorems 16.4 and 16.5 apply, by specializing the notions discussed in §11. This can be done in the following way.[16]

F being a family of sets, a biunique function f is called a ONE-TO-ONE TRANSFORMATION IN F if f maps every set in F which is included in $D(f)$ onto another set in F, and if the same applies to the inverse function f^{-1}. G being a set of one-to-one transformations in F, two sets A and B in F are said to be CONGRUENT UNDER G—in symbols,

$$A \underset{G}{\cong} B$$

—if there is a function f in G which maps A onto B. Analogously, we can define under what conditions the sets A and B are said to be EQUIVALENT BY FINITE, or INFINITE, DECOMPOSITION UNDER G; we use here decompositions of A and B into disjoint and pairwise congruent subsets (cf. the remarks in §11 which precede 11.20). The set G is called a QUASI-GROUP, or simply a GROUP, if it satisfies conditions (i)–(iii) of 11.18; we assume that the domain of the identity function in condition (i) is the union of all sets in F. G is called a COMMUTATIVE GROUP if, in addition, $fg = gf$ for any $f, g \,\varepsilon\, G$. The

[16] In connection with the following discussion compare Tarski [8].

set G is said to be FINITELY ADDITIVE if it satisfies the following condition: if f_0 and f_1 are two functions in G which map two disjoint sets A_0 and A_1 in F onto two other disjoint sets B_0 and B_1, then the function f defined by the formulas

$$D(f) = A_0 + A_1, \quad \text{and} \quad f(x) = f_i(x) \quad \text{for every} \quad x \in A_i, \quad i = 0, 1,$$

belongs to G also. In an analogous way we define what is meant by an INFINITELY ADDITIVE set of one-to-one transformations in F. As is easily seen, for every set (or group) G of one-to-one transformations in a countably complete set-field F there is a smallest finitely additive set (or group) H, and also a smallest infinitely additive set (or group) K, of one-to-one transformations in F which includes G; and, as in 11.23, we can show that the congruence under H (or K) coincides with the equivalence by finite (or infinite) decomposition under G.

The fundamental relations between these notions and those discussed in §11 are exhibited in the following:

LEMMA 16.6. *Let F be a countably complete set-field, and let G be a set of one-to-one transformations in F. For every function $f \in G$ and for every set $X \in \mathsf{F}$ which is included in $D(f)$, let $f^*(X)$ be the set onto which f maps X, i.e., the set of all elements $f(x)$ with $x \in X$; and let G^* be the set of all functions f^* thus defined. Then*

(i) *G^* is a set of partial automorphisms in the algebra $\langle \mathsf{F}, +, \sum \rangle$;*

(ii) *if G is a group, or a finitely (or infinitely) additive set, of one-to-one transformations in F, then G^* is a group, or a finitely (or infinitely) additive set, of partial automorphisms in $\langle \mathsf{F}, +, \sum \rangle$;*

(iii) *for any sets A, B, $\in \mathsf{F}$, we have $A \underset{G}{\approx} B$ if, and only if, $A \underset{G^*}{\approx} B$.*

PROOF: obvious, by the definitions of the notions involved (cf. the proof of 11.34).

We now have:

THEOREM 16.7. *If F is a countably complete set-field, and G is an infinitely additive group of one-to-one transformations in F, then the relation $\underset{G}{\approx}$ satisfies the hypothesis of 16.4, and hence also the conclusions of 16.4 and 16.5.*

PROOF: by 11.25, 16.2, and 16.4–16.6.

THEOREM 16.8. *If* F *is a countably complete set-field,* $\mathfrak{F} = \langle \mathsf{F}, \dotplus \rangle$ (*or* $\mathfrak{F} = \langle \mathsf{F}, \dotplus, \dot{\sum} \rangle$), *and* G *is a finitely additive group of one-to-one transformations in* F, *then*

(i) $\underset{G}{\approx}$ *is a finitely additive and finitely* (*as well as infinitely*) *refining equivalence relation in the algebra* $\mathfrak{F}$;

(ii) $\mathfrak{F}/\underset{G}{\approx}$ *is an R.A.*;

(iii) *conclusions* 16.5(i),(ii),(iii) *with* $n < \infty$, (vi), *and* (vii) *hold for the relation* $\underset{G}{\approx}$.[17]

PROOF: by 11.19, 11.24, 11.26, 11.28, 16.2, and 16.6; conclusion 16.5(vii) can be derived from 16.5(vi) in the way indicated in the proof of 16.5.

It may be mentioned that some of the results which apply to one-to-one transformations by 16.7 and 16.8 can be extended to many-to-one and even to many-to-many transformations; we are not, however, going to elaborate on this point.[18]

Further results applying to the algebra $\mathfrak{F}/\underset{G}{\approx}$ of 16.8 can be obtained by specializing assumptions which concern F or G. We are going to state some of these results, omitting partly their proofs. The methods of argument used in establishing the results in question are not very closely related to those applied in the present work, and the results themselves seem too special to be discussed in detail here.

THEOREM 16.9. *If* F *is the family of all subsets of a given set* A, $\mathfrak{F} = \langle \mathsf{F}, \dotplus \rangle$, *and* G *is a finitely additive group of one-to-one transformations in* F, *then Theorems* 2.11, 2.12, 2.15, 2.31–2.34, *as well as* 2.35 *and* 2.36 *with* $n < \infty$ *hold for all elements of the algebra* $\mathfrak{F}/\underset{G}{\approx}$ (*assuming that all the sums and multiples involved in these theorems are also in the algebra*); *and conclusions* 16.5(iv),(v) *apply to the relation* $\underset{G}{\approx}$.[19]

The proof that 16.5(v) applies to the relation $\underset{G}{\approx}$ and that, con-

[17] Theorem 16.5(i) in its application to the family F of all subsets of a set A and to the relation $\underset{G}{\approx}$ can be found—in a somewhat different form—in Banach [2]; conclusions (vi) and (vii) of 16.5, again in application to F and $\underset{G}{\approx}$, are stated (without proof) in Lindenbaum-Tarski [1], pp. 318 f.

[18] Cf. here Lindenbaum-Tarski [1], p. 316 ff., as well as Knaster [1] and Tarski [5].

[19] Theorem 16.5(v) for the relation $\underset{G}{\approx}$, and hence also Theorem 2.34 for the algebra $\mathfrak{F}/\underset{G}{\approx}$, has been established in König [1], pp. 129 ff. (where references to earlier papers on this subject by D. König and S. Valko can also be found); a simpler proof in the case $m = 2$ is given in Kuratowski [3]. Some of the remaining conclusions of 16.9 are mentioned in Lindenbaum-Tarski [1], p. 319.

sequently, 2.34 holds in the algebra $\mathfrak{F}/\underset{G}{\cong}$ of 16.9 is known from the literature; it is rather complicated, and it involves the axiom of choice in an essential way. By means of a similar method it can be shown that all other theorems of §2 listed in 16.9 hold in $\mathfrak{F}/\underset{G}{\cong}$. (It may be noticed that all these theorems can be derived arithmetically from the first part of 2.35 and from various elementary theorems which hold in $\mathfrak{F}/\underset{G}{\cong}$ by 11.28 and 16.8; they can also be derived from 2.33 in case $\mathfrak{F}/\underset{G}{\cong}$ is finitely closed.) From the fact that 2.12 holds in $\mathfrak{F}/\underset{G}{\cong}$ we conclude at once that 16.5(iv) applies to the relation $\underset{G}{\cong}$.

THEOREM 16.10. *Let* F *be an arbitrary set-field; let* $\mathfrak{F} = \langle \mathsf{F}, \dot{+} \rangle$; *let* G *be a commutative group of one-to-one transformations in* F, *and* H *the smallest finitely additive group of such transformations which includes* G. *We then have for all elements* a, b, *and* c *of the coset algebra* $\mathfrak{F}/\underset{H}{\cong}$ *(assuming that all the sums and multiples involved exist and are also in this algebra):*

(i) *if* $a + c = b + 2\cdot c$, *then* $a = b + c$;

(ii) *if* $n < \infty$ *and* $a + n\cdot c = b + (n + 1)\cdot c$, *then* $a = b + c$;

(iii) *if* $n < \infty$ *and* $b + (n + 1)\cdot c \leqq a + n\cdot c$, *then* $b + c \leqq a$;

(iv) *if* $a \neq 0$, *then* $a \neq 2\cdot a$ *(i.e., a is not idem-multiple), and—more generally—for no* $n < \infty$ *do we have* $n\cdot a = (n + 1)\cdot a$ *or* $(n + 1)\cdot a \leqq n\cdot a$;

(v) *Theorems* 2.10–2.15 *(again provided with suitable existential assumptions) hold for all elements of* $\mathfrak{F}/\underset{H}{\cong}$; *and* 16.5(iv) *applies to the relation* $\underset{H}{\cong}$.[20]

PROOF: We shall give here a rough outline of the proof of (i). The proof is not quite simple, but it has a constructive character (in the sense which will be discussed in the next section).

Given a set X and a function f, we shall denote by $f(X)$ the set of all function values $f(x)$ with $x \,\varepsilon\, X$. X is not supposed to be included in $D(f)$; if it is, $f(X)$ has the same meaning as $f^*(X)$ in 16.6.

By the hypothesis of (i), there is a set $D \,\varepsilon\, \mathsf{F}$ such that

$$(D/\underset{H}{\cong}) = a + c = b + 2\cdot c.$$

[20] Some of the conclusions of 16.10 are stated (without proof, and in a weaker form) in Tarski [9], pp. 222 f., and Tarski [1], p. 63. Conclusion (iv) of this theorem was originally obtained in collaboration with A. Lindenbaum (before other parts of the theorem were known).

Hence D can be represented in the form

(1) $D = A \dotplus C = B \dotplus C_1 \dotplus C_2$ where A, B, C, C_1, and C_2 are in F,

(2) $(A/\underset{H}{\cong}) = a, \quad (B/\underset{H}{\cong}) = b, \quad (C/\underset{H}{\cong}) = (C_1/\underset{H}{\cong}) = (C_2/\underset{H}{\cong}) = c,$

and consequently

(3) $$C \underset{H}{\cong} C_1 \quad \text{and} \quad C \underset{H}{\cong} C_2 .$$

As was mentioned before, congruence under H coincides with finite equivalence under G. Hence formulas (3) imply the existence of two numbers $r < \infty$ and $s < \infty$, two sequences of sets D_i and E_i, and a sequence of functions f_i which satisfy the conditions:

(4) $$C = \dot{\sum_{i<r}} D_i = \dot{\sum_{i<s}} D_{r+i}, \quad C_1 = \dot{\sum_{i<r}} E_i, \quad C_2 = \dot{\sum_{i<s}} E_{r+i},$$

(5) $D_i, E_i \,\varepsilon\, \mathsf{F}, \quad f_i \,\varepsilon\, G, \quad D_i \leqq D(f_i)$, and $f_i(D_i) = E_i$ for every $i < r + s$.

We can clearly assume that $r \neq 0$ and $s \neq 0$. By (4) and (5) there exist functions g_1 and g_2 defined by the formulas:

(6) $D(g_1) = D(g_2) = C, \quad g_1(x) = f_i(x)$ for $x \,\varepsilon\, D_i, \quad i < r$, and $g_2(x) = f_{r+i}(x)$ for $x \,\varepsilon\, D_{r+i}, i < s$.

Since H is a finitely additive group which includes G, and in view of (4)–(6), we have

(7) $$g_1, g_2 \,\varepsilon\, H, \quad D(g_1^{-1}) = C_1, \quad \text{and} \quad D(g_2^{-1}) = C_2 .$$

We define recursively an infinite sequence of functions h_n by putting

(8) $h_0 = i$ (the identity function whose domain is the union of all sets $X \,\varepsilon\, \mathsf{F}$);

(9) $$h_{2\cdot n+1} = h_n g_1 \quad \text{and} \quad h_{2\cdot n+2} = h_n g_2 \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

Hence, by (6) and (7),

(10) $$h_n \,\varepsilon\, H \quad \text{for} \quad n < \infty;$$

(11) $D(h_n) \leqq C$ and $D(h_n^{-1}) \leqq C_1 + C_2$ for $0 < n < \infty$.

X being an arbitrary set, we obtain by induction from (1), (6)–(9), and (11), without any special difficulty:

(12) if $t < \infty$ and $2^t - 1 \leqq n < p < 2^{t+1} - 1$, then

$$h_n(X) \cap h_p(X) = 0;$$

(13) if $n < p < \infty$, then $h_n(B) \cap h_p(X) = 0 = h_n^{-1}(A) \cap h_p^{-1}(X)$.

Furthermore, we put for every $n < \infty$

(14) $A'_n = A \cap h_n(B) - \sum_{m<n} h_n h_m^{-1}(A);$

(15) $A''_n = A \cap \sum_{m<n} h_n h_m^{-1}(A) - \sum_{m<n} \sum_{l<m} h_m h_l^{-1}(A);$

(16) $B'_n = h_n^{-1}(A) \cap B - \sum_{m<n} h_m^{-1}(A);$

(17) $B''_n = g_1 h_n^{-1}(A) \cap \sum_{m<n} g_1 h_m^{-1}(A) - \sum_{m<n} \sum_{l<m} g_1 h_n^{-1} h_m h_l^{-1}(A).$

From (1), (10), and (14)–(17) we see that

(18) $$A'_n,\ A''_n,\ B'_n,\ B''_n \ \varepsilon\ \mathsf{F}.$$

With the help of (6), (11), and (14)–(17), we obtain

$$B'_n \leqq D(h_n),\quad B''_n \leqq D(h_n g_1^{-1}),\quad h_n(B'_n) = A'_n, \quad \text{and} \quad h_n g_1^{-1}(B''_n) = A''_n;$$

therefore, in view of (7), (11), and (18),

(19) $$A'_n \underset{H}{\cong} B'_n \quad \text{and} \quad A''_n \underset{H}{\cong} B''_n \quad \text{for} \quad n=0, 1, 2, \cdots.$$

We also have by (1), (7), and (13)–(17)

(20) $A'_n \cap A'_p = A''_n \cap A''_p = B'_n \cap B'_p = B''_n \cap B''_p = 0$ for $n < p < \infty$,

(21) $$\sum_{n<\infty} A'_n \cap \sum_{n<\infty} A''_n = \sum_{n<\infty} B'_n \cap \sum_{p<\infty} B'_p = 0.$$

Now consider an integer $t > 0$ such that

(22) $$\frac{(t+1)\cdot(t+2)\cdot \cdots \cdot(t+r+s-1)}{1\cdot 2\cdot \cdots \cdot(r+s-1)} < 2^t.$$

The fact that such integers t exist is obvious. To fix the idea, we can assume that t is the smallest integer which satisfies (22). We put

(23) $$2^t - 1 = u \quad \text{and} \quad 2^{t+1} - 1 = v.$$

We are going to show that

$$C \leqq \sum_{m<u} h_m^{-1}(A). \tag{24}$$

In fact, suppose that x is an element in C which does not belong to any one of the sets $h_m^{-1}(A)$ for $m < u$. We conclude by an easy induction based upon (1), (6), (8), (9), and (23) that x belongs to all the domains $D(h_m)$ for $m < v$. By (12), the elements $h_m(x)$ for $m = u, u + 1, \cdots, v - 1$ are distinct from each other; their total number is, of course, 2^t. The group G being by hypothesis commutative, we see from (5), (6), (8), and (9) that each of these elements $h_m(x)$ is of the form

$$h_m(x) = f_1^{k_1} f_2^{k_2} \cdots f_{r+s}^{k_{r+s}}(x)$$

where $k_1, k_2, \cdots, k_{r+s}$ are non-negative integers satisfying the condition

$$\sum_{i<r+s} k_{i+1} = t. \tag{25}$$

Hence the number of distinct elements $h_m(x)$, $u \leqq m < v$, is at most equal to the number of different representations of the integer t in the form (25). As is easily seen, the latter number is

$$\frac{(t+1)\cdot(t+2)\cdot \cdots \cdot(t+r+s-1)}{1\cdot 2\cdot \cdots \cdot(r+s-1)},$$

so that consequently

$$2^t \leqq \frac{(t+1)\cdot(t+2)\cdot \cdots \cdot(t+r+s-1)}{1\cdot 2\cdot \cdots \cdot(r+s-1)}.$$

Since this clearly contradicts (22), our supposition proves to be wrong, and formula (24) is established.

(The fact that there is an integer u for which (24) holds is the only consequence of the commutativity of the group G that is involved in our proof. This remark paves the way for some possible extensions of 16.10.)

From (1) and (6)–(9) we see that

$$A \leqq h_0(B) + h_1(C) + h_2(C) \leqq h_0(B) + h_1(B) + h_2(B) + h_3(C) + \cdots + h_6(C);$$

by continuing this way we obtain in view of (23)

$$A \leqq \sum_{n<u} h_n(B) + \sum_{n<u+1} h_{u+n}(C).$$

Hence, by (23) and (24),

$$A \leqq \sum_{n<u} h_n(B) + \sum_{n<v}\sum_{m<n} h_n h_m^{-1}(A)$$

and

$$A = A \cap \sum_{n<u} h_n(B) + A \cap \sum_{n<v}\sum_{m<n} h_n h_m^{-1}(A). \tag{26}$$

Formulas (14), (15), and (26) easily imply

$$A = \sum_{n<u} A_n' + \sum_{n<v} A_n''. \tag{27}$$

(1), (6), (8), (23), and (24) give

$$A + C = B + C_1 + C_2 = \sum_{n<u} h_n^{-1}(A). \tag{28}$$

Consequently,

$$B = \sum_{n<u} h_n^{-1}(A) \cap B,$$

and hence, by (16),

$$B = \sum_{n<u} B_n'. \tag{29}$$

Furthermore, we establish the formula

$$C \leqq \sum_{n<v} \Big[h_n^{-1}(A) \cap \sum_{m<n} h_m^{-1}(A) - \sum_{m<n}\sum_{l<m} h_n^{-1}h_m h_l^{-1}(A)\Big]. \tag{30}$$

In fact, x being an element of C, the element $g_1(x)$ and $g_2(x)$ are in $A + C$ by (1), (6), and (7). Hence, by (24), there are integers $p < u$ and $q < u$ such that $h_p g_1(x)$ and $h_q g_2(x)$ are in A. We assume that p and q are the smallest integers of this sort. By (9),

$$h_p g_1(x) = h_{2\cdot p+1}(x) \quad \text{and} \quad h_q g_2(x) = h_{2\cdot q+2}(x).$$

Let n be the larger of the numbers $2\cdot p + 1$ and $2\cdot q + 2$; in view of (23), we have $n < v$. Clearly, x belongs to the set

$$h_n^{-1}(A) \cap \sum_{m<n} h_m^{-1}(A).$$

Moreover, we can show without major difficulties that $h_n(x)$ does not belong to any of the sets $h_m h_l^{-1}(A)$ with $l < m < n$; we make essential use here of (8) and (9) and of the way in which the number n has been determined. Consequently, x is in the set

$$h_n^{-1}(A) \cap \sum_{m<n} h_m^{-1}(A) - \sum_{m<n}\sum_{l<m} h_n^{-1}h_m h_l^{-1}(A)$$

where $n < v$; and since x is an arbitrary element of C, formula (30) proves to hold.

(6), (7), and (30) give

$$C_1 \leqq \sum_{n<v} [g_1 h_n^{-1}(A) \cap \sum_{m<n} g_1 h_m^{-1}(A) - \sum_{m<n} \sum_{l<m} g_1 h_n^{-1} h_m h_l^{-1}(A)].$$

The inclusion in the opposite direction being obvious by (7), we have in view of (6)

$$(31) \qquad C_1 = \sum_{n<v} B_n''.$$

By (1), (20), (21), (27), (29), and (31), we can write:

$$(32) \quad A = \dot{\sum_{n<u}} A_n' \dotplus \dot{\sum_{n<v}} A_n'' \quad \text{and} \quad B \dotplus C_1 = \dot{\sum_{n<u}} B_n' \dotplus \dot{\sum_{n<v}} B_n''.$$

Since, by 16.8, $\underset{H}{\cong}$ is a finitely additive relation in the algebra $\langle \mathsf{F}, \dotplus \rangle$, formulas (19) and (32) carry with them

$$A \underset{H}{\cong} B \dotplus C_1;$$

and, by (2), this implies at once the conclusion of (i):

$$a = b + c.$$

The proof of 16.10(i) has thus been completed. Conclusions (ii)–(iv) of 16.10 and Theorems 2.10–2.15 listed in (v) can easily be derived from (i), partly by induction; and 2.12 implies 16.5(iv) for the relation $\underset{H}{\cong}$.

Notice that the set-field F in 16.10 is not required to be countably complete, and also that conclusions 16.10(i)–(iv) do not apply to arbitrary C.A.'s.

The problem of measure discussed in §14 assumes a new aspect when applied to the algebras which we are now considering. Let F be a set-field, U an arbitrary set in F, and G a set of one-to-one transformations in F. A function f is called a (FINITELY ADDITIVE) MEASURE IN F NORMED BY U (or WITH THE UNIT U) AND INVARIANT UNDER G if it satisfies the following conditions:

(i) $D(f) = \mathsf{F}$, and $C(f)$ is a set of real numbers x, $0 \leqq x \leqq \infty$;

(ii) if A and B are two disjoint sets in F, then $f(A + B) = f(A) + f(B)$;

(iii) $f(U) = 1$;
(iv) if $A \in \mathsf{F}$ and $A \underset{G}{\cong} B$, then $f(A) = f(B)$.[21]
Similarly we define the notions of an INFINITELY ADDITIVE and of a STRICTLY POSITIVE measure in F (normed by U and invariant under G); cf. the remarks at the end of §14.

The new notion of measure is closely related to the one defined in 14.12. In fact, we have

LEMMA 16.11. *Let F be a set-field, U a set in F, and G a group of one-to-one transformations in F; moreover, let $\mathfrak{F} = \langle \mathsf{F}, \dot{+} \rangle$, let H be the smallest finitely additive group of one-to-one transformations in F which includes G, and let u be the coset of U under $\underset{H}{\cong}$. If f is a measure in F normed by U and invariant under G, then the formula*

$$\text{(i)} \qquad h(X/\underset{H}{\cong}) = f(X) \quad \textit{for every} \quad X \in \mathsf{F}$$

defines a function h which is a measure in the algebra $\mathfrak{F}/\underset{H}{\cong}$ normed by u (in the sense of 14.12). *If, conversely, h is such a measure, then the function f defined by formula* (i) *is a measure in F normed by U and invariant under G.*

PROOF: As was mentioned before, the relation between $\underset{G}{\cong}$ and $\underset{H}{\cong}$ is analogous to that between $\underset{G}{\simeq}$ and $\underset{H}{\simeq}$ described in 11.23; i.e., for any $A, B \in \mathsf{F}$, we have

$$A \underset{H}{\cong} B$$

if, and only if, there are sets $A_0, A_1, \cdots, A_i, \cdots, B_0, B_1, \cdots, B_i, \cdots \in \mathsf{F}$ such that

$$A = \dot{\sum_{i<n}} A_i, \quad B = \dot{\sum_{i<n}} B_i, \quad \text{and} \quad A_i \underset{G}{\cong} B_i \quad \text{for} \quad i < n < \infty.$$

Hence we easily see that, f being a measure in F normed by U and invariant under G, formula 16.11(i) defines indeed a function h which satisfies conditions 14.12(i)–(iii) with regard to the algebra $\mathfrak{F}/\underset{H}{\cong}$ and the element u; we apply here, of course, 6.3. The proof that the theorem also holds in the opposite direction is analogous, and even somewhat simpler.

A similar lemma applies to countably complete fields and infinitely additive measures.

[21] This notion of a measure was introduced (in application to the family of all subsets of a given set) in von Neumann [2], p. 78.

By means of 16.11 we can obtain a criterion for the existence of a measure in a given set-field F invariant under a given group of transformations G. To simplify the argument a little, assume F to be a countably complete set-field; we put

$$\mathfrak{F} = \langle \mathsf{F}, \dot{+} \rangle.$$

We construct the smallest finitely additive group H which includes G; U being the set of F chosen as the unit of measure, we put

$$u = (U/\overset{\sim}{\underset{H}{=}}).$$

By 16.8, $\mathfrak{F}/\overset{\sim}{\underset{H}{=}}$ is an R.A. It is not always finitely closed; however, we can imbed $\mathfrak{F}/\overset{\sim}{\underset{H}{=}}$ in a finitely closed R.A. $\mathfrak{A}'$ which is a finite closure of $\mathfrak{F}/\overset{\sim}{\underset{H}{=}}$ in the sense of 7.18. The procedure consists, roughly speaking, in extending F and G to a new countably complete set-field F' and a group of transformations G' in F' such that, for any $A, B \,\varepsilon\, \mathsf{F}'$, there are two disjoint sets A', $B' \,\varepsilon\, \mathsf{F}'$ which are congruent with A and B under G'. (Thus, the procedure does not have a purely algebraic character; we have not considered the problem whether every R.A. $\mathfrak{A}$ can be imbedded in a finitely closed R.A. $\mathfrak{A}'$ which is a finite closure of $\mathfrak{A}$.) By 13.9, $\mathfrak{A}'$ is a partially ordered groupoid with the refinement property. Hence, 14.13(ii) is a necessary and sufficient condition for the existence of a measure in $\mathfrak{A}'$ normed by u; and therefore it is also a sufficient condition for the existence of a measure in $\mathfrak{F}/\overset{\sim}{\underset{H}{=}}$. This condition involves multiples $n \cdot u$ and $(n + 1) \cdot u$ which are in $\mathfrak{A}'$ but not necessarily in $\mathfrak{F}/\overset{\sim}{\underset{H}{=}}$. However, by applying 2.1 with $p = n + 1 < \infty$, we can transform 14.13(ii) into an equivalent condition which involves only elements of $\mathfrak{F}/\overset{\sim}{\underset{H}{=}}$ (cf. 7.18):

(ii′) *there exists no integer $n < \infty$ and no double sequence of elements $u_{i,j}$ such that*

$$u = \sum_{j<n+1} u_{i,j} = \sum_{i<n} u_{i,j} \quad \text{for } i < n \quad \text{and} \quad j < n + 1.$$

Clearly, this condition is not only sufficient but also necessary for the existence of a measure in $\mathfrak{F}/\overset{\sim}{\underset{H}{=}}$ normed by u, and hence, by 16.11, for the existence of a measure in F normed by U and invariant under G. We can, of course, transform (ii′) into a condition involving only sets of F and congruence under G; but the final result of these transformations is rather complicated.

By applying, however, the same method of reasoning, we obtain

simple and interesting results if we subject F or $\dot{G}$ to additional assumptions in the way indicated in 16.9 and 16.10. In fact, we then have:

THEOREM 16.12.[22] *Let F be the family of all subsets of a given set A; let U be a subset of A, and G a group of one-to-one transformations in F. In order that there exist a finitely additive measure in F normed by U and invariant under G, it is necessary and sufficient that U satisfy the following condition:*

(i) *there are no disjoint sets U' and U'' such that*

$$U = U' + U'' \quad \textit{and} \quad U \underset{H}{\cong} U' \underset{H}{\cong} U''$$

where H is the smallest finitely additive group of one-to-one transformations in F which includes G;

or, in an equivalent formulation:

(ii) *there exist no two sequences of pairwise disjoint sets $V_0, V_1, \cdots, V_i, \cdots$ and $W_0, W_1, \cdots, W_i, \cdots$ with $i < n + p$, $n < \infty$, $p < \infty$, such that*

$$U = \sum_{i<n+p} V_i = \sum_{i<n} W_i = \sum_{i<p} W_{n+i}$$

$$\textit{and} \quad V_i \underset{G}{\cong} W_i \quad \textit{for every} \quad i < n + p.$$

We can describe briefly the content of 16.12(i) or 16.12(ii) by saying that the set U has NO PARADOXICAL DECOMPOSITION UNDER THE GROUP G.

THEOREM 16.13. *If F is an arbitrary set-field, U a non-empty set in F, and G a commutative group of one-to-one transformations in F, then there is always a measure in F normed by U and invariant under G.*

A detailed proof of 16.12 and 16.13 is superfluous; we apply here 13.19, 14.13, 14.14, and 16.8–16.10.

Theorem 16.13 can be extended to more comprehensive classes of groups of transformations; however, most generalizations in this direction cannot be obtained without imposing various restrictions on the set U, and—in opposition to 16.13—they are not implied by purely arithmetical properties of the algebra $\mathfrak{F}/\underset{H}{\cong}$ of 6.11.[23]

[22] This result was first stated in Tarski [4] and established in Tarski [1].

[23] Theorem 16.13, though in a less general form, can be found in von Neumann [2], pp. 79 and 94 f.; it was obtained as a generalization of an earlier result in Banach [1] concerning the existence of a measure on the Euclidean line. Compare these works of Banach and von Neumann for other results of a related nature.

The results stated in 16.9–16.13 can be given a more abstract form, in the style of §11; i.e., instead of a group G of one-to-one transformations in a set-field F, we can consider a group G of partial automorphisms in a G.C.A., or a Boolean algebra, $\mathfrak{A}$. We shall, of course, have to subject $\mathfrak{A}$ or G to some additional assumptions. In the case of 6.9 or 6.12 these assumptions must be strong enough to imply an isomorphism between $\mathfrak{A}$ and the algebra of all subsets of a given set; hence, we shall not achieve in this way any actual improvement of the results concerned. Theorems 16.10 and 16.13 in an abstract formulation apply to a commutative group of partial automophisms in an arbitrary Boolean algebra.

We now want to indicate briefly some applications of the results of this section to special set-fields.

Consider an arbitrary METRIC SPACE constituted by a set S and a DISTANCE FUNCTION d. Let F be the family of all subsets of S or, more generally, a countably complete set-field of such subsets which, together with any set A, contains also all sets which are CONGRUENT with A in the usual geometrical meaning. Furthermore, let G be the set of all ISOMETRIC (DISTANCE PRESERVING) TRANSFORMATIONS whose domain and counter-domain are included in S, although not necessarily identical with S. G is clearly a group of one-to-one transformations in F; and we can construct the smallest finitely additive group H and the smallest infinitely additive group K of one-to-one transformations in F, which include G.

We can now apply Theorem 16.7, or 16.8, taking K, or H, for G. Thus, F being the algebra of 16.4, $\mathfrak{F}/\underset{H}{\approx}$ is an R.A., and $\mathfrak{F}/\underset{K}{\approx}$ is a G.C.A. These algebras play an important role in connection with the geometric problem of measure in metric spaces. In fact, the ordinary GEOMETRIC MEASURE is a measure which is invariant under the group of all isometric transformations. Hence, by 6.11, the geometric problem of existence of a finitely additive measure in F is equivalent to the algebraic problem of existence of a measure in $\mathfrak{F}/\underset{H}{\approx}$ (in the sense of 14.12); and the same relation holds between the problems of an infinitely additive measure in F and in $\mathfrak{F}/\underset{K}{\approx}$. If, in particular, F is the family of all subsets of S, and U is a subset of S chosen as the unit of measure, then, by 16.12, the solution of the problem of a finitely additive measure is positive whenever U has no paradoxical decompositions (under G), and is negative otherwise.

We can take for G not the group of all isometric transformations, but a subgroup of it. Now, if G is commutative, e.g., if it is the group

of all TRANSLATIONS in a EUCLIDEAN SPACE, then, by 16.13, the solution of the problem of a finitely additive measure is always positive, whatever non-empty set U is chosen as the unit of measure. This result applies also to certain groups which are not commutative; it holds, for instance, if we take for S the EUCLIDEAN LINE, and for G the group of all isometric transformations in S (the SYMMETRIC REFLECTIONS included).

As another particular case, take for S the EUCLIDEAN n-DIMENSIONAL SPACE, for F the family of all subsets of S which are MEASURABLE IN THE SENSE OF LEBESGUE, and for G the set of all isometric transformations; K is again the smallest infinitely additive group which includes G. The algebra $\mathfrak{G} = (\mathsf{F}/\underset{K}{\cong})$ in this case is not only a G.C.A. but a C.A. as well (as can easily be shown by 5.25). The set Z of all elements of $\mathfrak{G}$ which are cosets of sets of measure 0 is clearly an ideal in $\mathfrak{G}$ in the sense of 9.1; hence, by 9.29, the coset algebra $\mathfrak{G}/\underset{Z}{=}$ is again a C.A. This coset algebra proves to be isomorphic with the algebra of non-negative real numbers of 14.2. The construction thus outlined can also be carried through in one step. In fact, consider the relation $\equiv$ which holds between two sets A and B in F if they are ALMOST CONGRUENT UNDER K; i.e. if there are sets A' and B' in F such that $A' \leqq A$, $B' \leqq B$, $A' \underset{K}{\cong} B'$, and the differences $A - A'$ and $B - B'$ are of measure 0. It can easily be shown that $\equiv$ satisfies the hypothesis of 16.4, so that $\mathfrak{F}/\equiv$ is a G.C.A. (and even a C.A.). Moreover, it turns out that $\equiv$ holds between two sets A and B if, and only if, they have the same measure.[24] Hence we conclude that $\mathfrak{F}/\equiv$ is isomorphic with the algebra of non-negative real numbers. This can serve as an illustration of the remarks previously made about the great variety of algebras which are isomorphic with the coset algebras of 16.4.

Now let S again be an arbitrary metric space; let B be the family of all BORELIAN SETS in S, and let G be the set of all biunique functions f which, together with their inverses, are MEASURABLE (B). B is by definition a countably complete set-field; and G can easily be shown to be an infinitely additive group of transformations in B. The relation $\underset{G}{\cong}$ is sometimes referred to as that of GENERALIZED HOMEOMORPHISM. By 16.7, the conclusions of 16.4 and 16.5 apply to the family B and to the relation $\underset{G}{\cong}$. Hence we obtain a great number of results concerning generalized homeomorphism. These

[24] Cf. Banach-Tarski [1], p. 277.

results, however, lose much of their significance when applied to Euclidean spaces or, more generally, to COMPLETE SEPARABLE SPACES; for the relation of generalized homeomorphism between Borelian sets then coincides with that of equality of power.[25] Nevertheless, certain consequences of these results may be of some interest even for Euclidean spaces. Consider, for instance, 16.5(vi). When applied to **B** and $\underset{G}{\cong}$, this result can be formulated as follows:

If A, A', B, C, and C' are sets in **B** *such that $A \leqq B \leqq C$ and $A' \leqq C'$, and if f and h are functions in G which map A onto A' and C onto C', respectively, then there is a set B' in* **B** *with $A' \leqq B' \leqq C'$, and a function g in G which maps B onto B'.*

Now assume that the BORELIAN CLASSES of all sets and functions which are involved in the hypothesis of this result are known; then, by analyzing the proof of 16.5(vi), we can evaluate the Borelian classes of the set B' and the function g involved in the conclusion.

[25] Cf. Kuratowski [1], in particular p. 208.

§17. ALGEBRA OF CARDINAL NUMBERS

As is well known, two sets A and B are said to be SET-THEORETICALLY EQUIVALENT (EQUINUMEROUS), or to have THE SAME POWER, if there is a biunique function f which maps A onto B.[26] Thus, the relation of set-theoretical equivalence is the congruence relation $\underset{G}{\cong}$ under the group G of all one-to-one transformations in the class S of all sets. Hence we obtain further

THEOREM 17.1. *The relation of set-theoretical equivalence coincides with the homogeneity relation* $\sim$ *in the algebra* $\mathfrak{S}$ *of* 16.1 (*and in every algebra* $\langle \mathsf{F}, +, \sum \rangle$ *where* F *is the family of all subsets of a given set* A).

PROOF: By taking in 16.6 the class S of all sets for F and the set of all one-to-one transformations in S for G, we conclude at once that two sets which are set-theoretically equivalent are homogeneous in the algebra $\mathfrak{S}$ in the sense of 11.29. The proof that the converse also holds is based upon 11.2–11.4, 11.30, 16.1, and 16.6, and is quite elementary.

In view of 17.1, we shall use the symbol '$\sim$' to denote the relation of set-theoretical equivalence.

THEOREM 17.2. *The hypothesis of* 16.4, *and hence also the conclusions of* 16.4 *and* 16.5, *are satisfied if we take for* F *the class* S *of all sets* (*or the family of all subsets of a given set* A), *and for* $\equiv$ *the relation* $\sim$ *of set-theoretical equivalence.*

PROOF: by 16.7 (with the substitutions indicated in the preceding proof for 16.6), or else by 11.32, 16.1, 16.4, 16.5, and 17.1.

Among the conclusions of 16.5 as applied to the relation $\sim$ we recognize certain familiar theorems of the theory of equality of power; thus, 16.5(i) and (ii) present two formulations of the CANTOR-BERNSTEIN EQUIVALENCE THEOREM, while 16.5(vi) has been called the SET-THEORETICAL MEAN VALUE THEOREM.[27]

[26] In connection with the following discussion consult Sierpiński [1].

[27] The problem of the origin of conclusions 16.5(i)–(viii) in their application to set-theoretical equivalence has been cleared up in §2—in footnotes to corresponding theorems on cardinal numbers. It should be added here that the mean value theorem is a translation of Theorem 2.27 (with $n = 1$) which was found by the author and stated in Lindenbaum-Tarski [1], p. 302; however, a particular case of this theorem—in fact, 16.5(vii)—was previously established in Korselt [1].

By the CARDINAL NUMBER (or the POWER) $\kappa(A)$ OF A SET A we understand the class of all sets X with $A \sim X$. We define the operation of ADDITION OF CARDINAL NUMBERS in such a way that

$$A = \dot{\sum_{i \varepsilon I}} A_i \quad \text{implies} \quad \kappa(A) = \sum_{i \varepsilon I} \kappa(A_i),$$

and in particular

$$A = B \dotplus C \quad \text{implies} \quad \kappa(A) = \kappa(B) + \kappa(C).$$

This operation of cardinal addition satisfies the unrestricted closure postulate and the general commutative and associative laws (cf. 12.3–12.5 where these laws have been formulated for the cardinal multiplication of isomorphism types). To show, e.g., that the closure postulate holds, consider any cardinals ν_i correlated with elements i of an arbitrary set I. Let A_i be any sets such that

$$\kappa(A_i) = \nu_i \quad \text{for every} \quad i \varepsilon I.$$

Given any element i in I, let B_i be the set of all couples $\langle i, x \rangle$ with $x \varepsilon A_i$; clearly

$$\kappa(B_i) = \kappa(A_i) = \nu_i \quad \text{for} \quad i \varepsilon I.$$

The union B of all these sets B_i is sometimes called the CARDINAL SUM OF THE SETS A_i.[28] Since any two sets B_i and B_j with $i \neq j$ are disjoint, we have

$$B = \dot{\sum_{i \varepsilon I}} B_i, \quad \text{and hence} \quad \kappa(B) = \sum_{i \varepsilon I} \nu_i .$$

Thus, the sum of arbitrarily many cardinals always exists.

The cardinal number of the empty set is denoted by '0'. Obviously, 0 is the class which consists of the empty set only. The fundamental properties of this number with regard to cardinal addition are entirely analogous to those of the isomorphism type 1 with regard to cardinal multiplication (cf. 12.7).

THEOREM 17.3. *K being the class of all cardinal numbers, the algebra* $\mathfrak{K} = \langle \mathrm{K}, +, \sum \rangle$ *(where* $+$ *and* $\sum$ *are the operations of cardinal addition) is a C.A.; the cardinal number* 0 *is the zero element of* K.

PROOF: As is easily seen (by 6.3), the algebra $\mathfrak{K}$ coincides with the coset algebra $\dot{\mathfrak{S}}/\sim$ where $\mathfrak{S}$ is the algebra of 16.1. Hence, by

[28] Cf. Whitehead-Russell [1], vol. 2, pp. 93 ff.

17.2 (16.4), $\mathfrak{K}$ is a G.C.A.; and since, as was pointed out, the operations $+$ and $\sum$ in $\mathfrak{K}$ satisfy the closure postulates, $\mathfrak{K}$ is also a C.A.

We can also prove 17.3 in another way. Let ν be any cardinal. We consider an arbitrary set A with $\kappa(A) = \nu$, and we denote by '$\varphi(\nu)$' the type of the Boolean algebra of all subsets of A under set-theoretical addition; $\varphi(\nu)$ does not depend on the choice of A. T being the class of all types $\varphi(\nu)$, we easily show that φ maps the algebra $\mathfrak{K}$ isomorphically onto the algebra $\langle \mathrm{T}, \times, \prod \rangle$; and since the latter algebra is a C.A. (this follows from 15.27), the same applies to the former.

As is well known from set theory, the algebra $\mathfrak{K}$ has certain very strong and general properties which by no means apply to arbitrary C.A.'s. They are stated in the following:

THEOREM 17.4. *The algebra* $\mathfrak{K}$ *of* 17.3 *is well ordered, and every infinite element in this algebra is idem-multiple.*

Compare here Definitions 4.1, 4.10, and 13.10.

THEOREM 17.5. *Let* Λ *be any non-empty class of cardinal numbers. For* $\mathfrak{L} = \langle \Lambda, +, \sum \rangle$ *to be a C.A., i.e., a cardinal subalgebra of the algebra* $\mathfrak{K}$ *of* 17.3, *it is necessary and sufficient that* $\mathfrak{L}$ *satisfy Postulate* 1.1.II *as well as the following condition:*

(i) *the class* Φ *of all finite cardinal numbers in* Λ *either is empty or consists of all finite multiples of a certain finite cardinal number* ν.

PROOF: Assume $\mathfrak{L}$ to be a C.A. Then 1.1.II certainly holds. Condition (i) is obviously satisfied if the class Φ either is empty or consists only of 0. Otherwise, let ν be the smallest cardinal in Φ which is different from 0. It is easily seen that Φ coincides with the class of all finite elements of Λ in the sense of 4.10, and that ν is indecomposable in the sense of 4.38. By 4.16, all multiples $m \cdot \nu$ with $m < \infty$ are in Φ. Now if μ is any element in Φ, we have $\mu \leqq m \cdot \nu$ for some $m < \infty$ (we can take, e.g., $m = \mu$ by identifying finite cardinals with finite integers); hence, by 4.44 (or 4.47), μ is a finite multiple of ν. Thus, $\mathfrak{L}$ satisfies condition (i).

Assume now that, conversely, $\mathfrak{L}$ satisfies Postulate 1.1.II and condition (i). The proof that the remaining postulates of 1.1 also hold in $\mathfrak{L}$ presents no difficulty to one familiar with the fundamental properties of cardinal numbers, and will not be carried through here in detail. We make an essential use of 17.4; it may also be useful to notice that, by (i), either Φ contains no number different from 0.

or else $\langle \Phi, +, \sum \rangle$ is a G.C.A. which is isomorphic with the algebra of finite integers (of 14.1).

THEOREM 17.6. *For a C.A.* $\mathfrak{A} = \langle A, +, \sum \rangle$ *to be isomorphic with a cardinal subalgebra of the algebra* $\mathfrak{K}$ *of* 17.3, *it is necessary and sufficient that* $\mathfrak{A}$ *be well ordered and that every infinite element in* A *be idem-multiple.*

PROOF: The necessity of the conditions follows directly from Theorem 17.4 and Definitions 4.1, 6.1, and 13.10. (4.4 and 17.5 are also useful here.) To show that they are sufficient, we arrange all elements of A in a transfinite sequence $a_0, a_1, \cdots, a_\xi, \cdots$ in such a way that $\xi < \eta$ implies $a_\xi \leqq a_\eta$ and $a_\xi \neq a_\eta$, the indices ξ, η, $\cdots$ being ORDINAL NUMBERS. We shall identify here finite ordinals and cardinals with finite integers; ω, $\omega + 1, \cdots, \xi, \cdots$ and $\aleph_0, \aleph_1, \cdots, \aleph_\xi, \cdots$ will be, as usual, the transfinite sequences of infinite ordinals and infinite cardinals in their natural order. We obviously have $a_0 = 0$; we can assume that a_0 is not the last term of the sequence. If a_1 is finite, we easily show that it is indecomposable, and that $a_\nu = \nu \cdot a_1$ for every $\nu < \omega$; we reason here by induction, and apply 4.38 and 4.44. We then correlate with every element a_ν, $\nu < \omega$, the finite cardinal ν, and with every element $a_{\omega+\xi}$ the infinite cardinal $\aleph_\xi$. In case a_1 is infinite, we correlate with every element a_ξ the cardinal $\aleph_\xi$. Λ being the class of all correlated cardinals, we show in either case without difficulty (by means of 6.1) that

$$\mathfrak{A} \cong \langle \Lambda, +, \sum \rangle.$$

Thus, our theorem holds in both directions.

As is seen from 17.6, the properties of the algebra $\mathfrak{K}$ expressed in Theorem 17.4 characterize the algebra $\mathfrak{K}$ and its cardinal subalgebras up to isomorphism. Theorem 17.4 obviously carries with it far-reaching simplifications in the arithmetic of cardinals. In consequence of this theorem, the theory of cardinal addition becomes trivial; and, in particular, the value of the result obtained in 17.3—i.e., of the fact that all the special theorems established in Part I apply to cardinal numbers—undergoes an essential depreciation.

It should be emphasized, however, that the properties of cardinals stated in 17.4 have been obtained in set theory with the essential

help of the axiom of choice in its most general form, and specifically with the help of the so-called well-ordering principle. On the other hand, the special theorems of Part I can be established for cardinal numbers either without any help of the axiom of choice, or by means of a very restricted form of this axiom; to derive these theorems, we do not have to use the general result stated in 17.3, but we can follow directly the lines of argument applied in Part I. Regarding this point, a close analysis of proofs leads to the following conclusions:

(1) All the theorems of Part I which involve only finitely many elements can be obtained for cardinal numbers without the help of the axiom of choice.

(2) The proof of the remaining theorems of Part I, which involve infinite sequences of elements (including those which involve infinite sequences of multiples of one element), requires the axiom of choice, but only in its application to denumerable families of sets. The axiom of choice in its general form occurs merely in the proof of Theorems 3.35 and 3.36 which involve arbitrary sets of elements.

Thus, for instance, theorems like 2.6, 2.10–2.15, 2.28 for $n < \infty$ and $p < \infty$, and 2.31–2.34 can be obtained for cardinal numbers without the help of the axiom of choice; although the proof of these theorems in their abstract algebraic form actually requires an application of this axiom (cf. the proofs of 2.6 and 2.31). Theorems involving infinite multiples of one element or of finitely many elements, like 1.29, 1.43, and 1.46, also belong here; to avoid the axiom of choice, they must be based, however, on an appropriate definition of an infinite multiple of a cardinal. (Given a cardinal ν and a set A with the power ν, we define $\infty \cdot \nu = \aleph_0 \cdot \nu$ as the power of the set B of all couples $\langle n, x \rangle$ with $n < \infty$ and $x \varepsilon A$.)

On the other hand, the proofs of theorems like 2.21 and 2.28 for $n = \infty$ or $p = \infty$ require a restricted application of the axiom of choice; 2.22 also belongs here since it involves an infinite sequence of multiples of one element. Even Postulate 1.1.II cannot be established for cardinal numbers without the help of the axiom of choice (restricted to denumerable families of sets).[29]

Analogous remarks apply to those theorems which do not belong to the arithmetic of cardinals proper, but refer directly to sets and to the relation of equality of power. Consider, for example, Theorem

[29] Cf. Sierpiński [1], pp. 118 ff.

16.5 which, as we know from 17.2, applies to the latter relation. Conclusions (i), (ii), (iii) for $n < \infty$, and (iv)–(vii) of this theorem can be obtained without the help of the axiom of choice; whereas conclusions (iii) for $n = \infty$ and (viii) require a restricted application of this axiom. Moreover, we can avoid the application of this axiom completely by means of a slight transformation of the theorems involved. We want to explain what we have in mind here by means of an example. Consider the following three theorems:

I. *If* $\mu_0, \mu_1, \cdots, \mu_i, \cdots$ *and* ν *are arbitrary cardinal numbers, and if*

$$\sum_{i<n} \mu_i \leqq \nu \quad \textit{for every} \quad n < \infty,$$

then

$$\sum_{i<\infty} \mu_i \leqq \nu.$$

II. *If* $A_0, A_1, \cdots, A_i, \cdots, B_0, B_1, \cdots, B_i, \cdots$, *and* C *are arbitrary sets such that*

$$\sum_{i<n} A_i \sim B_n \leqq C \quad \textit{for every} \quad n < \infty,$$

then there is a set B *such that*

$$\sum_{i<\infty} A_i \sim B \leqq C.$$

III. *If* $A_0, A_1, \cdots, A_i, \cdots$, *and* C *are arbitrary sets, and* $f_0, f_1, \cdots, f_i, \cdots$ *are biunique functions such that, for every* $n < \infty$, f_n *maps* $\sum_{i<n} A_i$ *onto a subset* B_n *of* C, *then there is a biunique function* f *which maps* $\sum_{i<n} A_i$ *onto a subset* B *of* C.

In I we recognize Theorem 2.21 applied to cardinals; II is the correlated theorem on equality of power, and is a particular case of 16.5(viii); III has been obtained from II by explicitly introducing functions which establish the equality of power.

A detailed analysis of the proof of 2.21—and of the theorems upon which 2.21 rests—shows that the axiom of choice can be avoided in the proof of III, and that the function f (and the set B) can be effectively constructed in terms of $A_0, A_1, \cdots, A_i, \cdots, C$, and $f_0, f_1, \cdots, f_i, \cdots$. II can be derived from III, and I from II. When deriving II from III, however, we must apply the axiom of choice

in order to correlate, with each couple of sets $\langle \sum_{i<n} A_i,\ B_n \rangle$, a biunique function f_n which maps one of these sets onto the other. Similarly, to derive I from II, we use this axiom when correlating, with each cardinal μ_n, a set A_n whose power is μ_n. Apart from these two steps, we can say that the proof of I and II has a constructive character. We see, moreover, that the application of the axiom of choice in the proof of I and II would be entirely superfluous if the theorems in question involved only a finite number of cardinals and sets. This explains the fact that theorems like 2.31–2.34 can be established for cardinal numbers without any help of the axiom of choice, although their proof in the general theory of C.A.'s seems to depend on this axiom in an essential way.

All the remarks made above can be extended to the congruence relations $\underset{G}{\cong}$ and the correlated coset algebras $\mathfrak{F}/\underset{G}{\cong}$ discussed in 16.7 (although this requires some further analysis of proofs involved); they do not, however, apply to arbitrary equivalence relations $\equiv$ and coset algebras $\mathfrak{F}/\equiv$ considered in 16.4 and 16.5. The question arises as to what the properties of the relations $\sim$ and $\underset{G}{\cong}$ are which make it possible to avoid the axiom of choice in proofs of various theorems involving these relations. An answer can be readily obtained by analyzing the proofs concerned; but we are not going to elaborate on this point.

Certain theorems of the arithmetic of cardinals are known whose proofs have a constructive character (in the sense roughly explained above), but which cannot be found among the results stated in Part I of this work. In the first place we have in mind here the following THEOREM ON THE DECOMPOSITION OF LINEAR FORMS:

Let $2 \leqq n \leqq \infty$; *let* $l_0, l_1, \cdots, l_i, \cdots$ *and* $m_0, m_1, \cdots, m_i, \cdots$ *be arbitrary integers* $(\leqq \infty)$, *and* $\nu_0, \nu_1, \cdots, \nu_i, \cdots$ *be arbitrary cardinals. If*

$$\sum_{i<n} l_i \cdot \nu_i = \sum_{i<n} m_i \cdot \nu_i,$$

then there are cardinals $\pi_0, \pi_1, \cdots, \pi_i, \cdots$ *and* $\rho_0, \rho_1, \cdots, \rho_i, \cdots$ *with the following properties:*

(i) $\sum_{i<n} l_i \cdot \pi_i = \sum_{i<n} m_i \cdot \pi_i$ *and* $\sum_{i<n} l_i \cdot \rho_i = \sum_{i<n} m_i \cdot \rho_i$;

(ii) $\nu_i = \pi_i + \rho_i$ *for every* $i < n$;

(iii) $\pi_0 \leqq \pi_1$ *and* $\rho_1 \leqq \rho_0$.

The proof of this theorem is not simple, but it requires an application of the axiom of choice only in case $n = \infty$ (and only to denumerable families of sets). The theorem is useful since it permits us to reduce certain problems regarding the relations between two given cardinals ν_0 and ν_1 to the case when these cardinals are comparable, i.e., when $\nu_0 \leqq \nu_1$ or $\nu_1 \leqq \nu_0$. (As is well known, the proof that the set of all cardinals is simply ordered, i.e., that any two cardinals are comparable, does not have a constructive character.) From the theorem on the decomposition of linear forms (with $n=3$) we can, for instance, derive in a simple way the cancellation laws for multiples, 2.31–2.34. The theorem can be extended to arbitrary coset algebras $\mathfrak{F}/\underset{G}{\cong}$ involved in 16.7. The problem remains open, however, whether the theorem can be derived from Postulates 1.1.I–VII alone, i.e., whether it holds in arbitrary C.A.'s; and the same applies to various particular cases and simple consequences of this theorem.

A general problem of a metamathematical nature arises in this connection. It is the problem of determining mutual relations between three sets of theorems on the addition of cardinals: the set of all theorems whose proof does not involve the axiom of choice (or requires only a restricted application of this axiom); the set of those theorems which apply to arbitrary coset algebras $\mathfrak{F}/\underset{G}{\cong}$ of 16.7 (and thus belong to the general theory of one-to-one transformations); and the set of those theorems which hold in arbitrary C.A.'s. It is obvious that certain theorems of the first set do not belong to the second; e.g., the theorem by which there is a smallest finite cardinal different from 0. These are, however, theorems of a very special nature; and it seems intuitively plausible that—apart from certain exceptional theorems—the first set is included in the second; it would be interesting to find a precise formulation and justification of this intuition. On the other hand, every theorem of the second set obviously belongs to the third set; and it seems very likely that the converse also holds. This conjecture would prove to be correct if we succeeded in establishing a general representation theorem by which every C.A. is isomorphic with one of the algebras $\mathfrak{F}/\underset{G}{\cong}$ involved in 16.7. It would also be interesting to solve the DECISION PROBLEM for the three sets of theorems under discussion; i.e., to find formal criteria which would enable us to decide in each particular case whether a given statement belongs to one of these three sets—

or else to show that no such criteria can be found. In any attempt to obtain a positive solution of this problem we should have to restrict ourselves to theorems which involve only binary addition.

The concluding remarks of this section concern the MULTIPLICATION OF CARDINALS. By the (STRONG) CARDINAL PRODUCT $\prod_{i \varepsilon I} A_i$ OF SETS A_i we understand the set F_0 of 6.11 (where we replace 'C_i' by 'A_i'), i.e., the set of all functions f with $D(f) = I$, and $f(i) \varepsilon A_i$ for every $i \varepsilon I$; in case I consists of two numbers, 0 and 1, the product is denoted by '$A_0 \times A_1$'. We define the (STRONG) PRODUCT OF CARDINAL NUMBERS in such a way that

$$\kappa(\prod_{i \varepsilon I} A_i) = \prod_{i \varepsilon I} \kappa(A_i),$$

and in particular

$$\kappa(A \times B) = \kappa(A) \times \kappa(B).$$

Cardinal numbers, however, do not constitute a C.A. under multiplication. In fact, Postulates 1.1.VI and VII fail here; this is easily seen by putting in 1.1.VI: $a = 3$, $b = 2^{\aleph_0}$, and $c_i = 2$ for $i < \infty$; and in 1.1.VII: $a_i = \aleph_0$ and $b_i = 2$ for $i < \infty$. On the other hand, we can consider WEAK PRODUCTS OF SETS AND CARDINALS by taking, instead of F_0, the set F_2 of 6.11. In this case, products of infinite cardinals coincide with their sums; and cardinal numbers form 'almost' a C.A. under weak multiplication, since Postulates 1.1.I–V and VII as well as the refinement theorem 2.3 are satisfied; the refinement postulate 1.1.VI still fails, however, as is seen by putting $a = 3$, $b = \aleph_0$, and $c_i = 2$ for $i < \infty$ (cf. an analogous remark in §14, after 14.9). Most theorems of Part I hold in the algebra of cardinals with weak multiplication; and this applies also to the algebra with strong multiplication, since finite products are the same in both algebras. In particular, both these algebras are finitely closed refinement algebras in the sense of 11.26.

The algebras of isomorphism types (of §12) and of cardinal numbers suggest the idea of studying abstract algebraic systems in which the fundamental operations are defined not only for couples or infinite sequences but also for sets of elements with higher powers. In particular, systems which satisfy the closure postulate, the general commutative and associative laws, and possibly also the general refinement theorem (12.14) seem to deserve some attention. It remains to be seen, of course, whether the study of such systems would bring fruitful results.

§ 18. ALGEBRA OF RELATION NUMBERS

By a (BINARY) RELATION R we understand an arbitrary SET OF ORDERED COUPLES. Instead of saying that the couple $\langle a, b\rangle$ is in R, we often say that R holds between a and b, in symbols,

$$a\,R\,b.$$

Since relations are sets, we can apply to them all theorems stated in §16. In particular, we have:

THEOREM 18.1. *The class* R *of all relations is a set-field which is countably complete (in the strict sense, and simply complete in the wider sense). All conclusions of* 16.1 *remain valid if we replace in them* 'S' *by* 'R'.

PROOF: obvious (with the help of 16.2).

We are now going to outline a construction which is specific for relations.

The set of all first terms occurring in ordered couples of a relation R is the DOMAIN OF R, $D(R)$; the set of all second terms is the COUNTER-DOMAIN (or CONVERSE DOMAIN) OF R, $C(R)$; and the union of $D(R)$ and $C(R)$ is the FIELD OF R, $F(R)$.[30] Two relations R and S are called STRICTLY DISJOINT if their fields are disjoint. By the STRICTLY DISJOINT SUM $\ddot{\sum}_{i\varepsilon I} R_i$ OF RELATIONS R_i we understand the union of these relations in case any two relations R_i and R_j with $i, j \varepsilon I$ and $i \neq j$ are strictly disjoint; otherwise, the operation $\ddot{\sum}$ is not defined. In particular, $R \ddot{+} S$ is the strictly disjoint sum of two relations R and S. By a PORTION OF A RELATION R we understand a relation S included in R (a SUBRELATION OF R) which is strictly disjoint with $R - S$; in other words, a relation S such that, for a certain T, $S \ddot{+} T = R$.

THEOREM 18.2. *If* F *is a countably complete set-field of relations —or, more generally, a family of relations such that* $\langle \mathsf{F}, +, \sum\rangle$ *is a G.C.A.* ($+$ *and* $\sum$ *being the operations of set-theoretical addition)— then* $\langle \mathsf{F}, \ddot{+}, \ddot{\sum}\rangle$ *is a multiple-free G.C.A.; the zero element of this algebra*

[30] Various notions applying to relations and involved in this section (domain, counter-domain, field, similarity, and relation number) have been introduced and studied in Whitehead-Russell [1], vol. 1, pp. 247 ff., and vol. 2, pp. 310 ff.

is the empty relation 0. *The conclusion applies, in particular, to the class* F = R *of all relations.*

PROOF: We show without difficulty that Postulates 5.1.I–V, with '$+$' and '$\sum$' replaced by '$\ddot{+}$' and '$\ddot{\sum}$', apply to relations of F, and that consequently the algebra $\langle \mathsf{F}, \ddot{+}, \ddot{\sum}\rangle$ is a G.C.A.; from 8.11 we see that this algebra is multiple-free. The last part of the theorem follows by 18.1.

THEOREM 18.3. *If* F *is the family of all portions of a relation* R, *then*

(i) F *is a complete set-field;*

(ii) *the G.C.A.'s* $\langle \mathsf{F}, \dot{+}, \dot{\sum}\rangle$ *and* $\langle \mathsf{F}, \ddot{+}, \ddot{\sum}\rangle$ *coincide.*

PROOF: (i) If the relations S_ι with $\iota \varepsilon I$ are portions of R, then $R - \sum_{i\varepsilon I} S_i$ is completely disjoint with each of the relations S_i, and therefore also with their sum; thus, $\sum_{i\varepsilon I} S_\iota$ and $R - \sum_{i\varepsilon I} S_i$ are again portions of R. Hence, as is easily seen, F is a complete set-field.

(ii) If S and T are two disjoint portions of R, they are strictly disjoint (since S is a subrelation of $R - T$); obviously, the converse also holds. Hence, and in view of 8.16 (and 18.2), the algebras in question coincide.

THEOREM 18.4. *If* F *is a countably complete set-field of relations, and if* $\equiv$ *is an equivalence relation which is infinitely additive and finitely refining in the algebra* $\mathfrak{F} = \langle \mathsf{F}, \ddot{+}, \ddot{\sum}\rangle$ *(i.e., if* $\equiv$ *satisfies conditions* 16.4(i),(ii) *with '$\dot{+}$' and '$\dot{\sum}$' changed to '$\ddot{+}$' and '$\ddot{\sum}$'), then the algebra* $\mathfrak{F}/\equiv$ *is a G.C.A.*

PROOF: by 6.10 and 18.2 (cf. also 6.4 and 6.7).

From 18.4 we can derive conclusions analogous to those stated in 16.5; the symbols '$\leqq$', '$+$', and '$\sum$' in these conclusions must be provided, however, with double dots. It is easily seen that the relations $\leqq$ and $\ddot{\leqq}$ in general do not coincide; the formula '$S \leqq R$' means that S is an arbitrary subrelation of R, while the formula '$S \ddot{\leqq} R$' means that S is a portion of R. On the other hand, in view of 18.3, the conclusions of 16.5 apply without any changes if we take for F the family of all portions of a given relation R.

Given a family F of relations, a biunique function f is called a SIMILARITY TRANSFORMATION IN F if, R being any relation in F whose field $F(R)$ is included in $D(f)$, the set of all ordered couples

$\langle f(x), f(y)\rangle$ with $\langle x, y\rangle \varepsilon R$ is again a relation in F; and if the same applies to the inverse function f^{-1}. We could now repeat the discussion of §16 which follows 16.5, by using similarity transformations instead of arbitrary one-to-one transformations, and strictly disjoint relations instead of simply disjoint sets. Thus, we could introduce the notions of CONGRUENCE of two relations UNDER A SET G OF SIMILARITY TRANSFORMATIONS, of a (QUASI-) GROUP and a FINITELY or INFINITELY ADDITIVE SET of such transformations; and we could obtain theorems entirely analogous to 16.6–16.8.

We confine ourselves here to the case when G is the class of all similarity transformations in the set-field R of all relations. Two relations R and S are called SIMILAR —in symbols,

$$R \simeq S$$

—if they are congruent under this class G, i.e., if there is a biunique function f which maps $F(R)$ onto $F(S)$ in such a way that $\langle x, y\rangle$ is in R if, and only if, $\langle f(x), f(y)\rangle$ is in S.

THEOREM 18.5. *The hypothesis and the conclusion of* 18.4 *are satisfied if we take for* F *the class* R *of all relations (or the family of all subrelations or of all portions of a given relation* R*), and for* $\equiv$ *the similarity relation* $\simeq$.

PROOF: We can derive this theorem by means of 11.25 from a lemma analogous to 16.6; a direct proof also presents no difficulty.

By the RELATION NUMBER $\rho(R)$ OF A RELATION R we understand the class of all relations which are similar to R. Two strictly disjoint sums $\ddot{\sum}_{i\varepsilon I} R_i$ and $\ddot{\sum}_{i\varepsilon I} S_i$ are clearly similar if the corresponding relations R_i and S_i are similar for every $i \varepsilon I$. Hence we can define the operation of CARDINAL ADDITION OF RELATION NUMBERS in such a way that

$$R = \ddot{\sum_{i\varepsilon I}} R_i \quad \text{implies} \quad \rho(R) = \sum_{i\varepsilon I} \rho(R_i),$$

and in particular

$$R = S \dot{+} T \quad \text{implies} \quad \rho(R) = \rho(S) + \rho(T).^{31}$$

This operation of cardinal addition satisfies the general closure postulate and the commutative and associative laws. To show that

[31] This operation has been defined in Birkhoff [1], p. 285, but only for numbers of partially ordering relations.

the closure postulate holds we proceed as in the case of cardinal numbers: Given any relation numbers σ_i with $i \varepsilon I$, we consider first arbitrary relations R_i such that

$$\rho(R_i) = \sigma_i \quad \text{for} \quad i \varepsilon I.$$

We then replace every relation R_i by the relation S_i consisting of all couples $\langle\langle i, x\rangle, \langle i, y\rangle\rangle$ where $\langle x, y\rangle \varepsilon R_i$; we then have clearly

$$\rho(S_i) = \rho(R_i) = \sigma_i \quad \text{for every} \quad i \varepsilon I.$$

Let S be the union of all these relations S_i; S is sometimes called the CARDINAL SUM OF THE (original) RELATIONS R_i. Since any two relations S_i and S_j with $i \neq j$ are strictly disjoint, we obtain

$$S = \sum_{i \varepsilon I}^{\cdot\cdot} S_i, \quad \text{and hence} \quad \rho(S) = \sum_{i \varepsilon I} \rho(S_i) = \sum_{i \varepsilon I} \sigma_i.$$

Thus, the cardinal sum of relation numbers σ_i exists.

The relation number 0 of the empty relation R clearly consists of R alone; thus, it coincides with the cardinal number 0, and has the same formal properties with regard to cardinal addition.

In case all relation numbers σ_i with $i \varepsilon I$ are equal to a given relation number σ, their sum depends merely on σ and on the power ν of the set I, and is called the ν^{th} MULTIPLE OF σ—in symbols,

$$\sum_{i \varepsilon I} \sigma_i = \nu \cdot \sigma.$$

These multiples satisfy the general distributive laws—with respect to sums of both cardinal numbers and relation numbers.

THEOREM 18.6. P *being the class of all relation numbers, the algebra* $\mathfrak{R} = \langle \mathrm{P}, +, \sum \rangle$ *(where* $+$ *and* $\sum$ *are the operations of cardinal addition) is a C.A.; the relation number* 0 *is the zero element of* $\mathfrak{R}$.

PROOF: analogous to that of 17.3, with 17.2 replaced by 18.5.

Thus, all theorems of Part I hold for cardinal sums of relation numbers. It is interesting to notice that many of these theorems also apply to ORDINAL SUMS OF RELATION NUMBERS—although the operation of ordinal addition in some respects differs fundamentally from that of cardinal addition (e.g., it is not commutative).[32]

[32] Compare theorems on ordinal types (i.e., on numbers of simply ordering relations) in Lindenbaum-Tarski [1], pp. 319 ff. Most of these theorems can be extended to arbitrary relation numbers. The author intends to outline an algebraic theory of ordinal addition in a special publication.

THEOREM 18.7.[33] *Every relation number σ can be represented in the form*

$$\sigma = \sum_{i \varepsilon I} \sigma_i$$

where all relation numbers σ_i with $i \varepsilon I$ are indecomposable in the algebra $\mathfrak{R}$ of 18.6. This representation is unique apart from order.

PROOF: Let S be a relation with

(1) $$\sigma = \rho(S).$$

By 18.3(i) the family F of all portions of S is a complete set-field. It is known that every set in a complete set-field F can be represented as a sum of disjoint sets which are indecomposable in the algebra $\langle \mathsf{F}, \dot{+}, \dot{\sum} \rangle$[34]; and, as is easily seen, two such representations can differ only in order (cf. the proof of 4.45). Hence, and in view of 18.3(ii), we have a unique representation of S:

(2) $$S = \ddot{\sum_{i \varepsilon I}} S_i$$

where the relations S_i with $i \varepsilon I$ are indecomposable in the algebra $\langle \mathsf{F}, \ddot{+}, \ddot{\sum} \rangle$. By passing to relation numbers, we obtain from (1) and (2)

(3) $$\sigma = \sum_{i \varepsilon I} \sigma_i \quad \text{where} \quad \sigma_i = \rho(S_i) \quad \text{for every} \quad i \varepsilon I.$$

Since the relations S_i are indecomposable, they are not empty (cf. 4.38 and 18.2); hence the relation numbers σ_i are different from 0. Assume that, for some $i \varepsilon I$,

(4) $$\sigma_i = \tau + \upsilon.$$

The relation of similarity $\simeq$ being finitely refining by 18.5, we obtain from (3) and (4), with the help of 6.7, a decomposition of S_i:

$$S_i = T \ddot{+} U \quad \text{where} \quad \tau = \rho(T) \quad \text{and} \quad \upsilon = \rho(U).$$

Hence,

$$S_i = T \quad \text{and} \quad \sigma_i = \tau, \quad \text{or} \quad S_i = U, \quad \text{and} \quad \sigma_i = \upsilon.$$

[33] Theorems 18.7 and 18.8 were obtained by B. Jónsson. Analogous results for ordinal addition had previously been established by Jónsson in collaboration with the author. The second part of 18.7 for numbers of partially ordering relations (and the general refinement theorem for sums of such numbers) can be found in Birkhoff [1], p. 286.

[34] This follows directly from Theorems 4 and 6 in Tarski [10], pp. 193 and 197.

Thus, σ_i is indecomposable; and (3) is a representation of σ as a sum of indecomposable relation numbers. In a similar way we derive the unicity of this representation from that of the representation of S in (2); we make use of the fact that the similarity relation is infinitely refining, not only in the sense of 6.7(ii) but also in a stronger sense applying to arbitrary sums. This completes the proof.

THEOREM 18.8. *$\mathfrak{K}$ being the algebra of cardinal numbers of* 17.3, *and $\mathfrak{R}$ the algebra of relation numbers of* 18.6, *we have*

$$\mathfrak{R} \cong \mathfrak{K}^{\mathrm{I}}_{s}$$

where I *is the class of all relation numbers which are indecomposable in* $\mathfrak{R}$.

PROOF: From 18.7, with the help of the general commutative and associative laws, we easily conclude that every relation number σ can be uniquely represented in the form

$$\sigma = \sum_{\tau \varepsilon I} [\varphi_\sigma(\tau) \cdot \tau]$$

where $\varphi_\sigma(\tau)$ are cardinals (not necessarily different from 0). Hence, φ_σ is a function whose domain is the class I of indecomposable relation numbers and whose counter-domain consists of cardinals. Thus, φ itself is a function which maps the class of all relation numbers σ onto the class of those functions which constitute the strong cardinal power $\mathfrak{K}^{\mathrm{I}}_{s}$ (cf. 6.11); and, with the help of the commutative and associative laws for sums and the distributive law for multiples of relation numbers, we easily show that φ establishes an isomorphism between $\mathfrak{R}$ and $\mathfrak{K}^{\mathrm{I}}_{s}$ (in the sense of 6.1).

Theorem 18.8 clearly remains true if $\mathfrak{K}$ and $\mathfrak{R}$ are regarded as algebras with unrestricted addition.

Theorems 18.7 and 18.8 improve considerably the result obtained in 18.6. In fact, by means of these theorems we can show directly that all the results of Part I hold in the arithmetic of relation numbers; the derivation is based upon elementary laws of the arithmetic of relation numbers (applied in the proof of 18.8) and fundamental properties of cardinals (e.g., those stated in 17.4), and is independent of the general theory of C.A.'s. Moreover, we can derive from 18.7 or 18.8 various properties of relation numbers which do not apply to arbitrary C.A.'s; for instance, we can show that every non-empty set of relation numbers has a greatest lower bound and, in case it is bounded above, also a least upper bound—in other words, that the

algebra of relation numbers is complete in the wider sense. As a consequence of 18.7 we obtain also the general refinement theorem for cardinal sums with an arbitrary number of terms. It should be noticed, however, that in the derivations based upon 18.7 and 18.8 we make an essential use of the axiom of choice in its most general form; whereas Theorem 18.6 permits us to extend the results of Part I to the arithmetic of relation numbers by means of arguments which have an essentially constructive character. It may also be mentioned that Theorem 18.6 can be extended to arbitrary coset algebras $\langle \mathsf{F}, \dot{+}, \ddot{\sum} \rangle / \underset{G}{\approx}$ where F is a set-field of relations and $\underset{G}{\approx}$ is the congruence relation under an infinitely additive group of similarity transformations in F; while 18.7 and 18.8 apply specifically to the algebra of relation numbers. Compare here the discussion of analogous problems in §17.

The results established in this section can be generalized considerably. They apply, practically without changes, to MANY-TERMED RELATIONS, i.e., to sets of n-termed sequences $\langle x_0, \cdots, x_i, \cdots \rangle$ with $i < n \leqq \infty$. They can easily be extended to FAMILIES OF SETS. For this purpose, we agree to call two families F and G of sets STRICTLY DISJOINT if the sums

$$\Sigma(\mathsf{F}) = \sum_{X \epsilon \mathsf{F}} X \quad \text{and} \quad \Sigma(\mathsf{G}) = \sum_{X \epsilon \mathsf{G}} X$$

are disjoint. We define the relation of DOUBLE EQUIVALENCE (or DOUBLE SIMILARITY) as the relation which holds between two families F and G if there is a biunique function f such that f maps $\Sigma(\mathsf{F})$ onto $\Sigma(\mathsf{G})$, and at the same time the function f^* defined as in 16.6 maps F onto G.[35] In terms of strict disjointness and double equivalence we define STRICTLY DISJOINT SUMS OF FAMILIES OF SETS, TYPES OF DOUBLE EQUIVALENCE OF FAMILIES, and CARDINAL SUMS OF TYPES; and we extend to these notions the results stated in 18.1–18.8.

In an entirely analogous way we can extend these results to ALGEBRAIC SYSTEMS wih one or more operations and to their ISOMORPHISM TYPES. It should be pointed out, however, that the notion of CARDINAL ADDITION has but a very small value for algebraic research. In fact, we are mostly interested in algebraic systems which are closed under their fundamental operations; it is easily seen, however, that the isomorphism type of a closed system is always indecomposable under cardinal addition, and hence is never a cardinal

[35] Cf. Whitehead-Russell [1], vol. 1, pp. 84 ff.

sum of two or more isomorphism types (unless all these types, with at most one exception, equal 0). For the same reason the operation of cardinal addition finds no applications in a discussion of such algebras which, like G.C.A.'s, are not necessarily closed, but are supposed to have a zero element; in connection with such algebras the related operation of CARDINAL MULTIPLICATION WITH THE RANK 2, which was briefly discussed at the end of §12, proves more useful.

The notions of set-theoretical equivalence of sets, similarity of relations, double equivalence of families of sets, and isomorphism of algebras can be regarded as particular cases of the GENERAL NOTION OF ISOMORPHISM, which is applicable to all possible kinds of mathematical systems. Consequently, cardinal numbers, relation numbers, types of double equivalence, and isomorphism types of algebras appear as particular cases of the GENERAL NOTION OF AN ISOMORPHISM TYPE. If we wished to give the results of this section the most general form, we should have to begin with a precise and adequate definition of the general notions just mentioned. The problem of formulating such a definition is, however, by no means simple; in trying to solve it we meet with various difficulties of a set-theoretical nature. We are not going to elaborate on this point.

Besides cardinal addition, we have come across another operation on isomorphism types—that of CARDINAL MULTIPLICATION. We have discussed in §12, and partly also in §§13 and 15, cardinal products of types of C.A.'s, G.C.A.'s, and related algebras; and in §17 we have made some remarks regarding products of cardinal numbers. The extension of the notion of a cardinal product to other isomorphism types, e.g., to relation numbers, presents no difficulty. From an algebraic point of view, the operation of cardinal multiplication is much more important than that of cardinal addition, and it has a much wider range of applications. We cannot expect, however, in the theory of this operation any general results analogous to 18.6–18.8. As was pointed out in §17, even the simplest isomorphism types, i.e., the cardinal numbers, do not constitute a C.A. under multiplication; compare here also Theorem 12.18 and the remarks which follow 12.21. On the other hand, we have obtained a number of rather interesting results applying to types of C.A.'s and of related algebraic systems (cf. 12.17, 12.21, 13.31, 15.27, and 15.28). We shall see in the appendix that these results can be extended to wider classes of algebras and that they have some implications for other kinds of isomorphism types, in fact, for relation numbers.

APPENDIX

CARDINAL PRODUCTS OF ISOMORPHISM TYPES

A. DIRECT PRODUCTS AND FACTOR ALGEBRAS

By an ALGEBRA we shall understand in this appendix an arbitrary system $\mathfrak{A} = \langle A, + \rangle$ constituted by a set A and a binary operation $+$, and subjected to one restriction only: A is supposed to contain the (uniquely determined) zero element 0 defined in 1.2.

An algebra $\mathfrak{B}$ formed by a subset B of A containing the element 0 and by the fundamental operation $+$ of $\mathfrak{A}$ will be called a SUBALGEBRA OF $\mathfrak{A}$, and also the set B itself will be referred to as a subalgebra of $\mathfrak{A}$ (or else a subalgebra of A under $+$); this somewhat restricts the usage of the term 'subalgebra' as determined in 6.13. In discussing subalgebras B, C, $\cdots$ of a given algebra $\mathfrak{A}$ we shall use the set-theoretical notation employed in §16; thus, for example, the formula '$B \leqq C$' will express the fact that the subalgebra B is included in the subalgebra C, and the symbolic expression '$B \cap C$' will denote the intersection of B and C. The subalgebra consisting only of 0 will be denoted by '$\{0\}$'.

Moreover, we shall apply to algebras $\mathfrak{A}$ various notions previously introduced in this work; the definitions stated in Parts I and II are understood to have been modified in the way indicated in 13.2.

We assume, in particular, that an isomorphism type $\tau(\mathfrak{A})$ has been correlated with every algebra $\mathfrak{A}$ in the way indicated in 12.1, and that the operation of cardinal multiplication has been defined both for algebras themselves and for their isomorphism types. As we know from 6.11 and 12.2 (compare also the remarks at the end of §12), several variants of this operation can be distinguished, which, however, have many properties in common. Thus, e.g., the cardinal multiplication of types satisfies the unrestricted closure postulate and the most general commutative and associative laws; cf. 12.3–12.5 and 13.29. In agreement with 12.6, the common type of all algebras $\mathfrak{A}$ which contain no element different from 0 will be called the UNIT TYPE 1; it has the properties indicated in 12.7. When applying definitions and elementary theorems on cardinal products, we shall not refer to them explicitly.

The factor relation between isomorphism types—which was mentioned a few times in §12—will now be defined in a formal way:

DEFINITION A.1. *A type α is called a* FACTOR OF A TYPE β *if $\alpha \leqq \beta$ in the algebra $\langle T, \times \rangle$ constituted by the class* T *of all isomorphism types and by the binary operation $\times$ of cardinal multiplication; i.e., if there is a type γ such that $\alpha \times \gamma = \beta$. The set of all factors of a type β is denoted by '$\Phi(\beta)$'.*

Various elementary properties of the factor relation, e.g., its reflexivity and transitivity, follow immediately from this definition and from fundamental properties of cardinal products. It may be mentioned that Definition 12.8 can now be reformulated in the following way:

COROLLARY A.2. *A type α is indecomposable if, and only if, $\Phi(\alpha)$ contains just two different types (in fact, the unit type* 1 *and α itself).*

We shall be concerned with various subalgebras of the algebra $\langle T, \times \rangle$ of A.1; and we shall be especially interested in subalgebras $\langle \Phi(\alpha), \times \rangle$.[1] Not much can be said in general about the algebras in question. The algebra $\langle T, \times \rangle$ is a groupoid; the unit type 1 is its zero element; and 1.33 holds in this algebra, i.e., $\alpha \times \beta = 1$ implies $\alpha = \beta = 1$. The same applies to algebras $\langle \Phi(\alpha), \times \rangle$—with the difference that they are not necessarily groupoids, but what we have called generalized groupoids (cf. the end of §13). The factor relation clearly coincides with the $\leqq$ relation, not only in the algebra $\langle T, \times \rangle$ but also in its subalgebras $\langle \Phi(\alpha), \times \rangle$; and the sets $\Phi(\alpha)$ are semi-ideals in this algebra—in fact, they coincide with the principal semi-ideals defined in 9.10.

The study of cardinal multiplication of algebras and isomorphism types reduces to a large extent to that of a certain operation on subalgebras of an algebra—an operation which will be referred to as DIRECT MULTIPLICATION.[2]

DEFINITION A.3. *Let $\mathfrak{A} = \langle A, + \rangle$ be an algebra, and let B_i be subalgebras of $\mathfrak{A}$ correlated with elements i of an arbitrary set I. By*

[1] The authors have included in this appendix only those results of their joint study of cardinal products which are more closely related to the main ideas and methods of the whole book. Some other results in this domain can be found in Jónsson-Tarski [1].

[2] The notions of a cardinal product of algebras and of a direct product of subalgebras are very often confused in the literature.

a DIRECT PRODUCT OF THE SUBALGEBRAS B_i *we understand a subalgebra* B *of* $\mathfrak{A}$ *satisfying the following condition:*

(i) *there is a function* f *which maps* $\langle B, +\rangle$ *isomorphically onto the cardinal product* $\prod_{i \varepsilon I} \langle B_i, +\rangle$ *in such a way that, for every* $i \varepsilon I$, B_i *consists of those elements* $x \varepsilon B$ *for which* $f_x(i) = x$, *while* $f_x(j) = 0$ *if* $j \varepsilon I$ *and* $i \neq j$.

In case the subalgebra B *is uniquely determined by this condition, we put*

$$B = \prod_{i \varepsilon I} B_i;$$

and if, in particular, I *consists of two numbers,* 0 *and* 1, $B_0 = C$, *and* $B_1 = D$, *we write*

$$B = C \times D.$$

Similarly we define a STRONG *and a* WEAK DIRECT PRODUCT OF SUBALGEBRAS, $\prod_{s,\, i \varepsilon I} B_i$ *and* $\prod_{w,\, i \varepsilon I} B_i$.

It should be remembered that the cardinal product $\prod_{i \varepsilon I} \langle B_i, +\rangle$ is constituted by functions g with

$$D(g) = I, \quad \text{and} \quad g(i) \varepsilon B_i \quad \text{for every} \quad i \varepsilon I$$

(cf. 6.11). Hence the function f considered in A.3(i) correlates, with every element $x \varepsilon A$, one of these functions $g = f_x$, so that

$$D(f_x) = I \quad \text{and} \quad f_x(i) \varepsilon B_i \quad \text{for every} \quad x \varepsilon A \quad \text{and} \quad i \varepsilon I.$$

The problem arises whether the subalgebra B in A.3 is indeed uniquely determined by condition (i) (provided such an algebra exists at all). We shall see in A.12 and A.15 that this is true in the case of a product of finitely many subalgebras and, more generally, in the case of a weak product of arbitrarily many subalgebras. It can be shown by means of an example that, in the case of plain and strong products, B is not, in general, uniquely determined by A.3(i). (To construct such an example, we can take for $\mathfrak{A}$ a group which is the cardinal product of an infinite sequence of groups of order 2.) Compare here, however, Theorem B.3 below.

The fundamental relation between cardinal products of types and direct products of subalgebras is exhibited in the following:

THEOREM A.4. *Let* $\mathfrak{A} = \langle A, +\rangle$ *be an algebra, and let* β_i *be isomorphism types correlated with elements* i *of a set* I. *In order that*

$$\tau(\mathfrak{A}) = \prod_{i \varepsilon I} \beta_i$$

it is necessary and sufficient that it be possible to correlate, with every $i \varepsilon I$, a subalgebra B_i of $\mathfrak{A}$ in such a way that A is a direct product of all these subalgebras and that

$$\tau(\langle B_\iota, +\rangle) = \beta_\iota \quad \textit{for every} \quad i \varepsilon I.$$

Similarly for strong and weak products.

PROOF: obvious.

In the following theorems we state various elementary properties of direct products, and in particular we formulate the general commutative and associative laws. The proofs, which are quite elementary, will be omitted.

THEOREM A.5. *Let, for every $i \varepsilon I$, B_ι be a subalgebra of an algebra $\mathfrak{A}$.*

(i) *If the set I is at most denumerable, then every direct product of subalgebras B_ι is also their strong direct product, and conversely.*

(ii) *If I is finite, the same applies to weak direct products.*

(iii) *If I consists of one element j, then*

$$\prod_{i \varepsilon I} B_i = B_j.$$

THEOREM A.6. *Let B_i for every $i \varepsilon I$ and C_j for every $j \varepsilon J$ be subalgebras of an algebra $\mathfrak{A}$. If there is a function f which maps I onto J in a one-to-one way so that $B_i = C_{f(i)}$ for every $i \varepsilon I$, then every direct product of subalgebras B_i is also a direct product of subalgebras C_j, and conversely. Similarly for strong and weak products.*

THEOREM A.7. *Let a set J_i be correlated with every element $i \varepsilon I$; let K be the set of all ordered couples $\langle i, j\rangle$ where $i \varepsilon I$ and $j \varepsilon J_i$; and let $B_{\iota,j}$ for every $\langle i, j\rangle \varepsilon K$ be a subalgebra of an algebra $\mathfrak{A}$. For B to be a direct product of all subalgebras $B_{i,j}$ it is necessary and sufficient that there exist subalgebras B_i of $\mathfrak{A}$ satisfying the following conditions: B is a direct product of all subalgebras B_i with $i \varepsilon I$; and, for every $i \varepsilon I$, B_i is a direct product of all subalgebras $B_{i,j}$ with $j \varepsilon J_i$. Similarly for strong and weak products.*

THEOREM A.8. *Let, for every $i \varepsilon I$, B_i be a subalgebra of an algebra $\mathfrak{A}$, and let B be a direct product of all subalgebras B_i with $i \varepsilon I$. If I'*

is a subset of I, then there is a subalgebra $B' \leqq B$ which is a direct product of all subalgebras B_i with $i \varepsilon I'$. Similarly for strong and weak products.

THEOREM A.9. *Let, for every $i \varepsilon I$, B_i be a subalgebra of an algebra $\mathfrak{A}$.*

(i) *If a direct product of all subalgebras B_i exists, then $B_i \cap B_j = \{0\}$ for all $i, j \varepsilon I$ with $i \neq j$.*

(ii) *If $\{0\}$ is a direct product of all subalgebras B_i, then $B_i = \{0\}$ for every $i \varepsilon I$. The converse implication also holds, and we have then*

$$\{0\} = \prod_{i \varepsilon I} B_i .$$

(iii) *If, for a given $j \varepsilon I$, B_j is a direct product of all algebras B_i, then $B_i = \{0\}$ for every $i \varepsilon I$ such that $i \neq j$. The converse implication also holds, and we have then*

$$B_j = \prod_{i \varepsilon I} B_i .$$

Similarly for strong and weak products.

THEOREM A.10. *Let, for every $i \varepsilon I$, B_i and C_i be subalgebras of an algebra $\mathfrak{A}$ such that $B_i \leqq C_i$. If C is a direct product of all subalgebras C_i, then there is a subalgebra $B \leqq C$ which is a direct product of all subalgebras B_i. Similarly for strong and weak products.*

We shall now concern ourselves with direct products of finite sequences of subalgebras. In this case the notion of a direct product can be characterized in terms of arithmetical properties of elements which constitute the subalgebras involved. In fact, we have:

THEOREM A.11. *Let $\mathfrak{A} = \langle A, + \rangle$ be an algebra, and let B, B_0, $B_1, \cdots, B_i, \cdots$ with $i < n < \infty$ be subalgebras of $\mathfrak{A}$. For B to be a direct product of $B_0, B_1, \cdots, B_i, \cdots$ with $i < n$, it is necessary and sufficient that the following conditions be satisfied:*

(i) *every element $b \varepsilon B$ can be represented in the form*

$$b = \sum_{i<n} b_i \quad \text{where} \quad b_i \varepsilon B_i \quad \text{for every} \quad i < n;$$

(ii) *if $b_i \varepsilon B_i$ for every $i < n$, then $\sum_{i<n} b_i$ exists and is in B;*

(iii) *if $b_i, b_i', b_i'' \varepsilon B_i$ for every $i < n$, we have*

$$\sum_{i<n} b_i = \sum_{i<n} b_i' + \sum_{i<n} b_i''$$

if, and only if, $b_i = b_i' + b_i''$ for every $i < n$.

PROOF: Assume conditions (i)–(iii) to be satisfied. By putting in (iii)

$$b_i'' = 0 \text{ for } i < n,$$

we see that every element b has only one representation (i). Thus, we can correlate with every $b \in B$ the unique sequence $b_0, b_1, \cdots, b_i, \cdots$ such that

$$b = \sum_{i<n} b_i.$$

From (i)–(iii) and 6.1 it is easily seen that this correlation establishes an isomorphism between $\langle B, +\rangle$ and $\prod_{i<n} \langle B_i, +\rangle$. If, furthermore, $j < n$, and $b_i = 0$ for $i < n$, $i \neq j$, then we have by 13.2

$$\sum_{i<n} b_i = b_j;$$

hence B_j consists of those elements $b \in B$ for which in the correlated sequence all terms equal 0 except, perhaps, the term b_j which equals b. Thus, according to A.3, B is a direct product of $B_0, B_1, \cdots, B_i, \cdots$.

Assume now, conversely, that B is a direct product of $B_0, B_1, \cdots, B_i, \cdots$. Let f be the function which maps $\langle B, +\rangle$ isomorphically onto $\prod_{i<n} \langle B_i, +\rangle$ in the way described in A.3(i). For every $i < n$ and $b \in B$ we put

$$b_i = f_b(i).$$

Thus, b_i is in B_i; and therefore

$$f_{b_i}(i) = b_i, \quad \text{and} \quad f_{b_i}(j) = 0 \quad \text{for} \quad j < n, \quad j \neq i.$$

Hence, by 13.2, we conclude that

$$f_b = \sum_{i<n} f_{b_i};$$

and consequently, by 6.1,

$$b = \sum_{i<n} b_i.$$

Condition A.11(i) has thus been established. In a similar way we show that conditions A.11(ii),(iii) are also satisfied; and our theorem proves to hold in both directions.

Corollary A.12. *If a subalgebra B of an algebra $\mathfrak{A}$ is a direct product of subalgebras $B_0, B_1, \cdots, B_i, \cdots$ with $i < n < \infty$, then it is the only direct product of these subalgebras, i.e.,*

$$B = \prod_{i<n} B_i.$$

Proof: by A.11(i),(ii) (and in view of A.3).

Theorem A.13. *Let, under the assumptions of* A.11,

$$B = \prod_{i<n} B_i.$$

We then have:

(i) *every element $b \in B$ has exactly one representation*

$$b = \sum_{i<n} b_i \quad \textit{with} \quad b_i \in B_i \quad \textit{for every} \quad i < n;$$

(ii) *if $b'_i, b''_i \in B_i$ for every $i < n$ and*

$$\sum_{i<n} b'_i + \sum_{i<n} b''_i \in A,$$

then $b'_i + b''_i \in B_i$ for every $i < n$;

(iii) *if $b'_i, b''_i, b'_i + b''_i \in B_i$ for every $i < n$, then*

$$\sum_{i<n} (b'_i + b''_i) = \sum_{i<n} b'_i + \sum_{i<n} b''_i;$$

(iv) *if $i < j < n$, $c \in A_i$, and $d \in A_j$, then $c + d = d + c$;*

(v) *if $i < n$, $j < n$, $k < n$, at least two of the three numbers i, j, and k are different, $c \in B_i$, $d \in B_j$, $e \in B_k$, $c + d \in B$, $c + e \in B$, and $d + e \in B$, then $c + (d + e) = c + (d + e)$.*

Proof: (i) follows from A.11(i),(iii). To derive (ii), we first apply A.11(i) to the element

$$b = \sum_{i<n} b'_i + \sum_{i<n} b''_i,$$

and then we make use of A.11(iii). Conclusion (iii) results directly from A.11(iii). By replacing in A.13(iii) most term by zeros, we arrive at (iv) (cf. the proof of 5.8). To obtain (v), we put

$$c = \sum_{l<n} c_l, \quad d = \sum_{l<n} d_l, \quad e = \sum_{l<n} e_l \tag{1}$$

where $c_i = c$, $d_j = d$, $e_k = e$, and all other terms are zeros. We observe that, for every $l < n$, at least one of the elements c_l, d_l,

and e_l is zero and all the elements c_l, d_l, e_l, $c_l + d_l$, $c_l + e_l$, and $d_l + e_l$ are in B; hence

(2) $$c_l + (d_l + e_l) = (c_l + d_l) + e_l \,\varepsilon\, B \quad \text{for} \quad l < n.$$

The conclusion of (v) easily follows from (1) and (2) by A.11(iii) and A.13(ii). The proof of A.13 is thus complete.

Various parts of A.13 can be generalized by an easy induction.

Conditions (i)–(iii) of Theorem A.11 can be modified in case we apply this theorem to a subalgebra B which is assumed to be closed under the operation $+$ (i.e., in which the sum $b_1 + b_2$ of any elements b_1, $b_2 \,\varepsilon\, B$ always exists and belongs to B). In fact, it is easily seen that in this case B is a direct product of subalgebras $B_0, B_1, \cdots, B_i, \cdots$ $(i < n < \infty)$ if, and only if, these subalgebras are included in B, are closed under $+$, and satisfy A.13(i),(iii). If, moreover, the associative law holds in the algebra $\mathfrak{A}$, condition A.13(iii) can be replaced by A.13(iv). Hence we see that Definition A.3 when applied to a finite sequence of subgroups of a group is equivalent to the usual definition of a direct product (or a direct sum).[3]

THEOREM A.14. *Let B as well as B_i for every $i \,\varepsilon\, I$ be subalgebras of an algebra $\mathfrak{A} = \langle A, +\rangle$.*

(i) *For B to be a weak direct product of all subalgebras B_i it is necessary and sufficient that, for every finite sequence $i_0, i_1, \cdots, i_k, \cdots$ with $k < n < \infty$ of distinct elements of I, the product $\prod_{k<n} B_{i_k}$ exist, and that B be the union of all these finite products.*

(ii) *In case I is the set of all finite integers, this necessary and sufficient condition can be expressed as follows: all products $\prod_{i<n} B_i$ with $n < \infty$ exist, and*

$$B = \bigcup_{n<\infty} \prod_{i<n} B_i.$$

PROOF: (i) If B is a weak product of subalgebras B_i, then, by A.8, all finite products $\prod_{k<n} B_{i_k}$ exist and are subsets of B. Now consider a function f which maps $\langle B, +\rangle$ onto $\prod^{w}_{i\varepsilon I} \langle B_i, +\rangle$ in the way described in A.3(i). Thus, for every $b \,\varepsilon\, B$, f_b is a function with

$$D(f_b) = I, \quad \text{and} \quad f_b(i) = B_i \quad \text{for} \quad i < \infty.$$

[3] Cf., for instance, van der Waerden [1], pp. 141 ff.

Moreover, there are only finitely many elements $i_0, i_1, \cdots, i_k, \cdots$ in I, $k < n < \infty$, for which $f_b(i_k) \neq 0$. As is easily seen, we have

$$f_b = \sum_{k<n} f_{b_{i_k}};$$

hence, by isomorphism,

$$b = \sum_{k<n} b_{i_k}; \tag{1}$$

and therefore, by A.11(ii), b belongs to $\prod_{k<n} B_{i_k}$. Consequently, B is the union of all such finite products; and the condition of our theorem proves to be necessary.

Assume now, conversely, that all finite products $\prod_{k<n} B_{i_k}$ exist and that B is their union. Consider any elements $b \;\varepsilon\; B$ and $i \;\varepsilon\; I$. As is easily seen, b belongs to a certain product $\prod_{k<n} B_{i_k}$ where $i_k = i$ for some $k < n$; thus, by A.11(i), b has a representation (1). If now b belongs to another product $\prod_{l<p} B_{j_l}$ where $j_l = i$ for some $l < p$, and the corresponding representation is

$$b = \sum_{l<p} b'_j,$$

then we can arrange all elements i_k and j_l in one sequence, construct the corresponding product, and show by means of A.13(i),(iv),(v) that

$$b_i = b_{i_k} = b'_{j_l}.$$

Thus, the element b_i is uniquely determined by b and i; and we can correlate with b the function f_b defined by the formula

$$f_b(i) = b_i.$$

The proof that f satisfies condition A.3(i), with '$\prod$' changed to '$\prod_w$', is quite elementary; we apply various parts of A.11 and A.13. Thus, according to A.3, B is a weak direct product of subalgebras B_i.

(ii) follows from (i) by A.8; and the proof of A.14 is complete.

From A.11 and A.14 we can easily obtain a characterization of a weak direct product of subalgebras B_i in terms of arithmetical properties of elements belonging to these subalgebras. (For G.C.A.'s this has been implicitly done in 12.13.)

COROLLARY A.15. *If a subalgebra B of an algebra $\mathfrak{A}$ is a weak direct product of subalgebras B_i of $\mathfrak{A}$ with $i \varepsilon I$, then B is the only weak direct product of these subalgebras, i.e.,*

$$B = \prod_{i \varepsilon I}{}_w B_i.$$

PROOF: by A.14 (cf. A.3).

DEFINITION A.16. *Let $\mathfrak{A} = \langle A, + \rangle$ be an algebra.*

(i) *A subalgebra B of $\mathfrak{A}$ is called a* (DIRECT) FACTOR OF *$\mathfrak{A}$ if there is a subalgebra C of $\mathfrak{A}$ such that $A = B \times C$.*

(ii) *The family of all factors of $\mathfrak{A}$ will be denoted by '$\mathsf{F}(\mathfrak{A})$'. The algebras $\mathfrak{F}(\mathfrak{A}) = \langle \mathsf{F}(\mathfrak{A}), \times \rangle$ as well as $\langle \mathsf{F}(\mathfrak{A}), \times, \prod \rangle$ and $\langle \mathsf{F}(\mathfrak{A}), \times, \prod_w \rangle$ will be referred to as* FACTOR ALGEBRAS OF *$\mathfrak{A}$. The symbols '$\leqq_f$', '$\mathsf{\Pi}_f$', '$\cap_f$', 'U_f', and '$\cup_f$', will be used to denote the $\leqq$ relation (or* FACTOR RELATION*) and the operations of forming the least upper bound (or* LEAST COMMON MULTIPLE*) and the greatest lower bound (or* GREATEST COMMON FACTOR*) in these factor algebras.*

It should be noticed that the operations $\prod$ and $\prod_w$ in factor algebras are understood to be restricted to infinite sequences of factors; cf. the remark which follows 8.7. Hence, in view of A.5, the factor algebras with the operation $\prod_s$ need not be discussed separately. A fundamental relation between factor algebras and the algebras of isomorphism types $\langle \Phi(\alpha), \times \rangle$ and $\langle \Phi(\alpha), \times, \prod_w \rangle$ will be established in A.18.

THEOREM A.17. *Let $\mathfrak{A} = \langle A, + \rangle$ be an algebra, and β an isomorphism type. For β to be a factor of $\tau(\mathfrak{A})$ it is necessary and sufficient that there be a (direct) factor B of $\mathfrak{A}$ such that*

$$\beta = \tau(\langle B, + \rangle).$$

PROOF: by A.1, A.3, A.4, A.12, and A.16.

THEOREM A.18. *For every algebra $\mathfrak{A}$ we have:*

(i) *the isomorphism relation $\cong$ between factors of $\mathfrak{A}$ is a finitely refining and finitely additive equivalence relation in the algebra $\mathfrak{F}(\mathfrak{A})$, which in addition is infinitely additive in the algebras $\langle \mathsf{F}(\mathfrak{A}), \times, \prod \rangle$ and $\langle \mathsf{F}(\mathfrak{A}), \times, \prod_w \rangle$;*

(ii) *if $\tau(\mathfrak{A}) = \alpha$, then the algebras $\langle \Phi(\alpha), \times \rangle$ and $\langle \Phi(\alpha), \times, \prod_w \rangle$ are isomorphic with the coset algebras $\mathfrak{F}(\mathfrak{A})/\cong$ and $\langle \mathsf{F}(\mathfrak{A}), \times, \prod_w \rangle/\cong$, respectively.*

PROOF: (i) follows easily from A.4 with the help of 6.1, 6.4, 6.7, 6.13, A.3, A.12, and A.16. The proof of (ii) is analogous to that of 12.15 or 12.19; we apply A.4 and A.12 again, as well as 6.3, A.15, and A.17.

Since we use a multiplicative notation for factor algebras, it would be more proper to speak of (finitely and infinitely) multiplicative relations in these algebras. We prefer, however, to use the term 'additive relation,' so as not to deviate from the terminology employed in the main body of the work.

It may be noticed that the relation $\cong$ discussed in A.18(i) is not only finitely but also infinitely refining in the algebra $\langle \mathsf{F}(\mathfrak{A}), \times, \prod_w \rangle$. It is doubtful, however, whether this applies also to the algebra $\langle \mathsf{F}(\mathfrak{A}), \times, \prod \rangle$, for a direct product of infinitely many factors is not always uniquely determined. For the same reason A.18(ii) cannot be extended to algebras $\langle \mathsf{F}(\mathfrak{A}), \times, \prod \rangle$.

THEOREM A.19. *If a factor B of an algebra A is a direct product of subalgebras B_i of $\mathfrak{A}$ with $i \varepsilon I$, then all these subalgebras are factors of $\mathfrak{A}$. Similarly for strong and weak products.*

PROOF: by A.7, A.12, and A.16.

THEOREM A.20. *Let B be a factor of an algebra $\mathfrak{A} = \langle A, + \rangle$. If any two of the three elements b_1, b_2, $b_1 + b_2 \varepsilon A$ are in B, then the third element is also in B.*

PROOF: By A.16 we have for some subalgebra C of $\mathfrak{A}$

$$A = B \times C.$$

If $b_1, b_2 \varepsilon B$, we apply A.13(ii) with $n = 2$ by putting

$$b_0' = b_1, \quad b_0'' = b_2, \quad b_1' = b_1'' = 0,$$

and we obtain $b_1 + b_2 \varepsilon B$. If b_1, $b_1 + b_2 \varepsilon B$, we get by A.11(i)

$$b_2 = b + c \quad \text{where} \quad b \varepsilon B \quad \text{and} \quad c \varepsilon C;$$

hence

$$(b_1 + b_2) + 0 = (b_1 + 0) + (b + c);$$

therefore, by A.11(iii),

$$0 = 0 + c = c \quad \text{and} \quad b_2 = b + 0 = b;$$

so that finally $b_2 \varepsilon A$. In a similar way we show that $b_2, b_1 + b_2 \varepsilon B$ implies $b_1 \varepsilon B$.

THEOREM A.21. *Let B, C, $B \times C$ be factors of an algebra $\mathfrak{A} = \langle A, + \rangle$, and let $b, b' \varepsilon B$ and $c, c' \varepsilon C$. We then have:*

(i) *$b' + c' \leqq b + c$ if, and only if, $b' \leqq b$ and $c' \leqq c$;*

(ii) *if $x \varepsilon A$, $x \leqq b$, and $x \leqq c$, then $x \leqq 0$;*

(iii) *if $x \varepsilon A$, $b \leqq x$, and $c \leqq x$, then $b + c \leqq x$.*

Hence, in case the algebra $\mathfrak{A}$ is partially ordered, we have

(iv) *$b \cap c = 0$ and $b \cup c = b + c$.*

PROOF: By A.16 there is a subalgebra D such that

(1) $$A = (B \times C) \times D.$$

Suppose

(2) $$b' + c' \leqq b + c.$$

We then have by 11.5 for some element $a \varepsilon A$

$$b + c = (b' + c') + a.$$

By A.11(ii) and A.20, a is in $B \times C$, and hence by A.11(i) there are elements $b'' \varepsilon B$ and $c'' \varepsilon C$ such that

$$a = b'' + c''.$$

Therefore

(3) $$b + c = (b' + c') + (b'' + c''),$$

and by A.11(iii)

(4) $$b = b' + b'' \quad \text{and} \quad c = c' + c''.$$

Hence

(5) $$b' \leqq b \quad \text{and} \quad c' \leqq c.$$

If, conversely, (5) is true, then (4) holds for some b'', $c'' \varepsilon A$, and by A.20 we conclude that $b'' \varepsilon B$ and $c'' \varepsilon C$. Therefore (3) holds by A.11(iii), and (2) follows directly from (3). Thus (i) has been established in both directions.

Suppose now $x \varepsilon A$. Let, by A.11(i) and (1),

(6) $$x = (b_1 + c_1) + d \quad \text{where} \quad b_1 \varepsilon B, \quad c_1 \varepsilon C, \quad d \varepsilon D.$$

Under the hypothesis of (ii), we have

$$(b_1 + c_1) + d \leqq (b + 0) + 0 \quad \text{and} \quad (b_1 + c_1) + d \leqq (0 + c) + 0.$$

By now applying (i) several times, we obtain first, with the help of A.11(ii),

$$b_1 + c_1 \leqq b + 0, \quad b_1 + c_1 \leqq 0 + c, \quad \text{and} \quad d \leqq 0;$$

hence

$$c_1 \leqq 0, \quad b_1 \leqq 0, \quad \text{and} \quad d \leqq 0;$$

and finally

$$(b_1 + c_1) + d \leqq (0 + 0) + 0, \quad \text{i.e.,} \quad x \leqq 0.$$

Similarly, under the hypothesis of (iii), we obtain from (6) by (i)

$$b \leqq b_1, \quad c \leqq c_1, \quad 0 \leqq d,$$

and consequently

$$b + c \leqq (b_1 + c_1) + d, \quad \text{i.e.,} \quad b + c \leqq x.$$

We have, moreover, by 1.2, 1.5, and A.13(iv),

$$0 \leqq b \leqq b + c \quad \text{and} \quad 0 \leqq c \leqq b + c.$$

Hence, if $\mathfrak{A}$ is partially ordered and therefore the relation $\leqq$ is antisymmetric (cf. 13.10), we can apply Definitions 3.1 and 3.2, and thus replace conditions A.21(ii),(iii) by formulas (iv). This completes the proof.

It may be noticed that the $\leqq$ relation between elements of an algebra $\mathfrak{A}$, as defined in 1.5, does not play any distinguished role in the discussion of algebraic systems of an arbitrary nature. There are several other relations which deserve equal attention and to which Theorem A.21 can be extended; for instance, the relations $\leqq_1$ and $\leqq_2$ defined in the following way:

$$a \leqq_1 b \quad \textit{if} \quad a + c = c + a = b \quad \textit{for some } c;$$

$$a \leqq_2 b \quad \textit{if} \quad (c + a) + d = b \quad \textit{for some } c \textit{ and } d.$$

In this work, however, we are interested almost exclusively in algebras which satisfy the commutative and associative laws; and in these algebras the relations in question coincide.

Various elementary properties of factor algebras can easily be derived from general theorems on direct products, A.6–A.10. Notice in particular

THEOREM A.22. *For every algebra* $\mathfrak{A} = \langle A, +\rangle$ *we have:*

(i) *if* $B, C, B \times C \in \mathsf{F}(\mathfrak{A})$, *then* $B \times C = C \times B$;

(ii) *if* $B, C, D \in \mathsf{F}(\mathfrak{A})$ *and either* $B \times C, (B \times C) \times D \in \mathsf{F}(\mathfrak{A})$ *or* $C \times D, B \times (C \times D) \in \mathsf{F}(\mathfrak{A})$, *then all these products are in* $\mathsf{F}(\mathfrak{A})$ *and* $(B \times C) \times D = B \times (C \times D)$;

(iii) $\{0\} \in \mathsf{F}(\mathfrak{A})$, *and* $B \times \{0\} = \{0\} \times B = B$ *for every* $B \in \mathsf{F}(\mathfrak{A})$; *i.e.,* $\{0\}$ *is the zero element of* $\mathfrak{F}(\mathfrak{A})$;

(iv) *if* $B, B \times B \in \mathsf{F}(\mathfrak{A})$, *then* $B = \{0\}$; *i.e., the algebra* $\mathfrak{F}(\mathfrak{A})$ *is multiple-free;*

(v) $A \in \mathsf{F}(\mathfrak{A})$; *and for every* $B \in \mathsf{F}(\mathfrak{A})$ *we have* $B \leqq_f A$, *i.e., there is a* $C \in \mathsf{F}(\mathfrak{A})$ *such that* $A = B \times C$.

PROOF: (i) and (ii) follow easily from A.6, A.7, and A.16. (iii) is an immediate consequence of A.9(iii) and A.16, while (iv) results at once from A.9(i) and 8.11. Finally, (v) follows from A.6, A.9(iii), and A.16.

The following theorem is of a somewhat less obvious nature and has many important consequences:

THEOREM A.23 (MODULAR LAW). *Let* $B, C, B \times C$, *and* D *be factors of an algebra* $\mathfrak{A} = \langle A, +\rangle$. *If* $B \leqq D$, *then*

$$(B \times C) \cap D = B \times (C \cap D).$$

PROOF: By A.10, A.12, and A.16, the product $B \times (C \cap D)$ exists and

(1) $$B \times (C \cap D) \leqq B \times C.$$

Since

$$B \leqq D \text{ and } C \cap D \leqq D,$$

we easily see by A.11(i) and A.20 that every element of $B \times (C \cap D)$ is in D, i.e.,

$$B \times (C \cap D) \leqq D.$$

Hence and by (1)

(2) $$B \times (C \cap D) \leqq (B \times C) \cap D.$$

Consider now any element a in $(B \times C) \cap D$. By A.11(i) we have

$$a = b + c \quad \text{with} \quad b \in B \quad \text{and} \quad c \in C.$$

Since

$$B \leqq D,$$

b is in D. But a is also in D. Consequently, by A.20, c is in D and therefore in $C \cap D$. Hence, by A.11(ii), a is in $B \times (C \cap D)$. Thus

$$(3) \qquad (B \times C) \cap D \leqq B \times (C \cap D).$$

(2) and (3) at once give the conclusion.

Theorem A.23 can be generalized. In fact, B and C can be assumed to be arbitrary subsets of A; $B \times C$ can be interpreted as what is sometimes called the COMPLEX SUM OF B AND C, i.e., as the set of all elements $b + c \in A$ with $b \in B$ and $c \in C$ (so that the existence of $B \times C$ does not have to be postulated), and the same applies to $B \times (C \cap D)$; finally, we can assume that D is an arbitrary subset of A which has the properties stated in A.20 for B.[4]

THEOREM A.24. *Let B and C be factors of an algebra $\mathfrak{A} = \langle A, + \rangle$. We then have $B \leqq C$ if, and only if, $B \leqq_f C$ in the algebra $\mathfrak{F}(\mathfrak{A})$, i.e., if $B \times D = C$ for some $D \in \mathsf{F}(\mathfrak{A})$.*

PROOF: By A.22(v) we have

$$(1) \qquad A = B \times D' \quad \text{for some} \quad D' \in \mathsf{F}(\mathfrak{A}).$$

Now if

$$(2) \qquad B \leqq C,$$

we obtain, by applying A.23 (with $C = D'$ and $D = C$),

$$(B \times D') \cap C = B \times (D' \cap C),$$

and hence, by (1),

$$C = B \times (D' \cap C).$$

Thus, by A.19, the set $D' \cap C$ is a factor of $\mathfrak{A}$, and we have

[4] In connection with the modular law cf. Birkhoff [3], pp. 34 ff., where bibliographical references to earlier publications can be found. For normal subgroups of a group this law was discovered by R. Dedekind. In Baer [1], p. 455, the modular law is stated for a class of algebras with one operation which is more comprehensive than that of groups.

(3) $$B \leqq_f C.$$

Conversely, (3) implies (2) by A.3. The theorem thus holds in both directions.

COROLLARY A.25. *For every algebra $\mathfrak{A}$, the algebra $\mathfrak{F}(\mathfrak{A})$ is partially ordered.*

PROOF: by A.24.

This corollary can also be easily derived from A.9(iii), A.16, and A.22(ii),(iii).

COROLLARY A.26. *Let $\mathfrak{A}$ be an arbitrary algebra.*

(i) *If B_i for every $i \varepsilon I$ as well as $\bigcap_{i \varepsilon I} B_i$, or $\bigcup_{i \varepsilon I} B_i$, are factors of $\mathfrak{A}$, then*

$$\bigcap_{i \varepsilon I} B_i = \bigcap_{i \varepsilon I}{}_f\, B_i, \quad \text{or} \quad \bigcup_{i \varepsilon I} B_i = \bigcup_{i \varepsilon I}{}_f\, B_i .$$

(ii) *If $B \times C \varepsilon \mathsf{F}(\mathfrak{A})$, then $B \cap C = B \cap_f C = \{0\}$.*

(iii) *If $B \times C \varepsilon \mathsf{F}(\mathfrak{A})$, then $B \times C = B \cup_f C$.*

PROOF: (i) follows directly from A.24 (and 3.1, or 3.2). (ii) results from (i), A.9(i), and A.22(iii). Furthermore, we have by A.3

$$B \leqq B \times C \quad \text{and} \quad C \leqq B \times C;$$

and we easily see from A.11(i) and A.20 that

$$B \leqq X \quad \text{and} \quad C \leqq X \quad \text{imply} \quad B \times C \leqq X$$

for every factor X. Hence, by A.24, we obtain (iii).

THEOREM A.27. *If $B, C, D, B \times C$, and $(B \times C) \times D$ are factors of an algebra $\mathfrak{A}$, then*

$$(B \times C) \cap (B \times D) = (B \times C) \cap_f (B \times D) = B.$$

PROOF: By A.22(i),(ii), the product $B \times D$ exists and is a factor of $\mathfrak{A}$. By A.3 we have

$$B \leqq B \times C.$$

We now apply A.23 with 'C' and 'D' replaced by 'D' and '$B \times C$', respectively; and we obtain

(1) $$(B \times D) \cap (B \times C) = B \times [D \cap (B \times C)].$$

Since $(B \times C) \times D$ is a factor of $\mathfrak{A}$, we have by A.26(ii)

$$(2) \qquad D \cap (B \times C) = \{0\}.$$

The conclusion follows from (1) and (2) by A.22(iii) and A.26(i).

Corollary A.26(iii) can be generalized:

THEOREM A.28. *If B_i for every $i \varepsilon I$ and $\prod_{i \varepsilon I}^{w} B_i$ are factors of an algebra $\mathfrak{A}$, then*

$$\prod_{i \varepsilon I}^{w} B_i = \bigcup_{i \varepsilon I}^{f} B_i .$$

PROOF: From A.7, A.14, and A.19 we see that the products $\prod_{k<n} B_{i_k}$ are factors of $\mathfrak{A}$ for every finite sequence $i_0, i_1, \cdots, i_k, \cdots \varepsilon I$. By A.14 and A.26(i),(iii), we easily conclude that $\prod_{i \varepsilon I}^{w} B_i$ is the least upper bound of all such finite products in the algebra $\mathfrak{F}(\mathfrak{A})$, and that each of these finite products is the least upper bound of its factors. Hence the conclusion follows at once.

As a further application of the modular law, we may mention that Theorem 2.18(i) holds in every factor algebra $\langle \mathsf{F}(\mathfrak{A}), \times, \prod^{w} \rangle$ and applies to an arbitrary (even non-denumerable) system of factors.

B. ALGEBRAS WHOSE FACTOR ALGEBRAS ARE BOOLEAN

The theorems established in the preceding section provide us with much information about factor algebras. In particular, we see from A.22 that the algebra $\mathfrak{F}(\mathfrak{A})$ is a multiple-free generalized groupoid in which all elements have a common upper bound; thus, this algebra has many properties in common with disjunctive Boolean algebras defined in 15.14. All these results, however, do not seem to have any interesting consequences for the algebras of types $\langle \Phi(\alpha), \times \rangle$, in which we are primarily interested. To obtain such consequences, we restrict ourselves in our further discussion to those algebras $\mathfrak{A}$ for which the factor algebra is a disjunctive Boolean algebra.

THEOREM B.1. *For every algebra* $\mathfrak{A} = \langle A, + \rangle$ *the following conditions are equivalent:*

(i) $\mathfrak{F}(\mathfrak{A})$ *is a disjunctive Boolean algebra;*

(ii) $\mathfrak{F}(\mathfrak{A})$ *has the refinement property.*

PROOF: From A.22 we see that the factor algebra $\mathfrak{F}(\mathfrak{A})$ has all the properties of disjunctive Boolean algebras listed in 15.14, except perhaps the refinement property. Hence the conclusion.

Several other conditions are known which are equivalent to those of B.1. We list here the following:

(iii) $\langle \mathsf{F}(\mathfrak{A}), \mathsf{U}_f \rangle$ *is a Boolean algebra;*

(iv) *if* $A = B_1 \times B_2 = C_1 \times C_2$, *then there are factors* D_1, D_2, D_3, *and* D_4 *of* $\mathfrak{A}$ *such that* $B_1 = D_1 \times D_2$, $B_2 = D_3 \times D_4$, $C_1 = D_1 \times D_3$, *and* $C_2 = D_2 \times D_4$;

(v) *if* $B \varepsilon \mathsf{F}(\mathfrak{A})$ *and* $A = C_1 \times C_2$, *then there are factors* B_1 *and* B_2 *of* $\mathfrak{A}$ *such that* $B = B_1 \times B_2$, $B_1 \leqq C_1$, *and* $B_2 \leqq C_2$;

(vi) *if* $B, C_1 \times C_2 \varepsilon \mathsf{F}(\mathfrak{A})$, *then* $B \cap (C_1 \times C_2) = (B \cap C_1) \times (B \cap C_2)$.

In view of A.24, condition (iii) can also be put in the following form:

$\mathsf{F}(\mathfrak{A})$ *is a Boolean algebra under set-theoretical inclusion*

(cf. the beginning of §15). The proof that conditions (iii)–(vi) follow from B.1(i) is based upon elementary properties of disjunctive Boolean algebras, which we assume here to be known. In deriving (iii) we use 15.16; (iv) is but a particular case of B.1(ii) (cf. 13.18); in deriving (v) and (vi), we apply A.24 and A.26(i). The proof that each of the conditions (iii)–(v) implies B.1(i) is more involved and will not be given here; it is essentially based upon the modular law A.23. Condition (vi), which will be generalized in Theorem B.4 below, clearly implies B.1(ii).

In our further discussion we shall become acquainted with comprehensive classes of algebras $\mathfrak{A}$ which satisfy conditions B.1(i), i.e., for which the factor algebra $\mathfrak{F}(\mathfrak{A})$ is a disjunctive Boolean algebra. As a simple example of an algebra which does not satisfy this condition we may mention the four group, i.e., a group which is the direct product of two subgroups of order 2.

THEOREM B.2. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be an algebra which satisfies* B.1(i). *If* $B, C \in \mathsf{F}(\mathfrak{A})$, *then* $B \cap C \in \mathsf{F}(\mathfrak{A})$ *and* $B \cap C = B \cap_f C$.

PROOF: By A.22(v), we have for some factors B' and C' of $\mathfrak{A}$

$$(1) \qquad A = B \times B' = C \times C'.$$

Hence, by B.1 (and 13.18), there are factors D_1, D_2, D_3, and D_4 such that

$$(2) \quad B = D_1 \times D_2, \quad C = D_1 \times D_3, \quad B' = D_3 \times D_4, \quad \text{and} \quad C' = D_2 \times D_4.$$

From (1) and (2) we easily see by A.22(ii) that $(D_1 \times D_2) \times D_3$ is a factor. Consequently, by A.27,

$$(3) \qquad (D_1 \times D_2) \cap (D_1 \times D_3) = (D_1 \times D_2) \cap_f (D_1 \times D_3) = D_1 .$$

(2) and (3) imply the conclusion at once.

For algebras $\mathfrak{A}$ which satisfy condition B.1(i) the results obtained in A.15 and A.28 for weak direct products can be extended to direct products and strong direct products; it must be assumed, however, that the products involved not only exist but are factors of $\mathfrak{A}$. In fact, we have

THEOREM B.3. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be an algebra which satisfies* B.1(i). *If a factor* B *of* $\mathfrak{A}$ *is a direct product of factors* B_i *of* $\mathfrak{A}$ *with* $i \varepsilon I$, *then*

(i) $B = \bigcup_{f,\, i \varepsilon I} B_i$;

(ii) B *is the only direct product of factors* B_i, *i.e.*,

$$B = \prod_{i \varepsilon I} B_i .$$

Similarly for strong products.

PROOF: By A.3 and A.24 we have

(1) $$B_i \leqq_f B \quad \text{for every} \quad i \varepsilon I.$$

Consider any factor X satisfying the condition

(2) $$B_i \leqq_f X \quad \text{for every} \quad i \varepsilon I.$$

Since $\mathfrak{F}(\mathfrak{A})$ is a disjunctive Boolean algebra, the factor $B \cap_f X$ exists, and we have for some factor C

(3) $$B = (B \cap_f X) \times C.$$

Since (1) and (2) give

$$B_i \leqq_f B \cap_f X,$$

there are factors D_i such that

$$B \cap_f X = B_i \times D_i ,$$

and hence, by (3) and A.22(ii),

(4) $$B = B_i \times (D_i \times C) \quad \text{for every} \quad i \varepsilon I.$$

B being by hypothesis a direct product of factors B_i, let f be the function with the properties described in A.3(i). Thus, f maps $\langle B, + \rangle$ isomorphically onto $\prod_{i \varepsilon I} \langle B_i , + \rangle$; and, for any given $i \varepsilon I$, B_i consists of all elements $x \varepsilon B$ such that

$$f_x(i) = x, \quad \text{and} \quad f_x(j) = 0 \quad \text{for} \quad j \varepsilon I, \quad j \neq i.$$

Let C_i be the set of all elements $x \varepsilon B$ for which

$$f_x(i) = 0.$$

We easily verify, e.g., by means of A.11 and A.12 that

(5) $$B = B_i \times C_i \quad \text{for every} \quad i \varepsilon I.$$

Consequently, by A.19, the sets C_i are factors of $\mathfrak{A}$. Since the cancellation law holds in disjunctive Boolean algebras, (4) and (5) imply

$$C_i = D_i \times C \quad \text{for every} \quad i \varepsilon I.$$

Hence, by A.3, every element $x \varepsilon C$ belongs to all the sets C_i and therefore

$$f_x(i) = 0 \quad \text{for every} \quad i \varepsilon I.$$

This shows that f_x is the zero element of the algebra $\prod_{i\varepsilon I} \langle B_i, + \rangle$; and therefore, by isomorphism, $x = 0$. Thus,

(6) $$C = \{0\}.$$

From (3) and (6) we obtain by A.22(iii)

(7) $$B = B \cap_f X \leqq_f X$$

Since (1) holds, and since (2) implies (7) for every factor X, we have by 3.2 and A.25

$$B = \bigcup_{i\varepsilon I}{}_f\, B_i.$$

The second part of the conclusion follows at once by A.3.

One of the consequences of A.28 and B.3 is that $\prod_{i\varepsilon I}^{w} B_i$ and $\prod_{i\varepsilon I} B_i$ (or $\prod_{i\varepsilon I}^{s} B_i$) cannot both be factors of an algebra $\mathfrak{A}$ satisfying B.1(i), unless there are only finitely many elements $i \varepsilon I$ for which $B_i \neq \{0\}$. As further consequences of these results we obtain the general distributive law for direct multiplication with respect to the operation $\cap$ (or $\cap_f$), as well as the general refinement theorem for products of arbitrarily many factors:

Theorem B.4. *Let $\mathfrak{A}$ be an algebra which satisfies* B.1(i), *and let B as well as C_i for every $i \varepsilon I$ be factors of $\mathfrak{A}$. If $\prod_{i\varepsilon I} C_i$ is also a factor of $\mathfrak{A}$, then*

$$B \cap \prod_{i\varepsilon I} C_i = \prod_{i\varepsilon I} (B \cap C_i).$$

Similarly for strong and weak products.

Proof: By B.3 we have

(1) $$\prod_{i\varepsilon I} C_i = \bigcup_{i\varepsilon I}{}_f\, C_i.$$

Thus, $\bigcup_{f\, i \varepsilon I} C_i$ exists in $\mathfrak{F}(\mathfrak{A})$. Hence, in view of the fact that every disjunctive Boolean algebra is completely distributive (cf. 15.4 and 15.20), we obtain

$$(2) \qquad B \cap_f \bigcup_{f\, i \varepsilon I} C_i = \bigcup_{f\, i \varepsilon I} (B \cap_f C_i).$$

There are certainly factors D_i such that

$$C_i = (B \cap_f C_i) \times D_i \quad \text{for every} \quad i \varepsilon I.$$

Hence, by A.7, A.19, and B.3(ii),

$$\prod_{i \varepsilon I} C_i = \prod_{i \varepsilon I} (B \cap_f C_i) \times \prod_{i \varepsilon I} D_i .$$

Here $\prod_{i \varepsilon I} (B \cap_f C_i)$ is a factor, and therefore, by B.3, we have

$$(3) \qquad \prod_{i \varepsilon I} (B \cap_f C_i) = \bigcup_{f\, i \varepsilon I} (B \cap_f C_i).$$

The conclusion follows immediately from (1)–(3) by B.2.

Corollary B.5. *Let $\mathfrak{A}$ be an algebra which satisfies* B.1(i); *and let B_i for every $i \varepsilon I$ and C_j for every $j \varepsilon J$ be factors of $\mathfrak{A}$. If $\prod_{i \varepsilon I} B_i$ is also a factor of $\mathfrak{A}$ and if*

$$\prod_{i \varepsilon I} B_i = \prod_{j \varepsilon J} C_j ,$$

then

$$B_i = \prod_{j \varepsilon J} (B_i \cap C_j) \text{ and } C_j = \prod_{i \varepsilon I} (B_i \cap C_j) \text{ for every } i \varepsilon I \text{ and } j \varepsilon J,$$

where all the sets $B_i \cap C_j$ are factors of $\mathfrak{A}$. Similarly for strong and weak products.

Proof: By B.4, with the help of A.3 and A.19.

Notice that in B.5 we have not only extended the refinement property to arbitrary products, but also determined the nature of factors occurring in 'refining' decompositions.

B.5 can easily be carried over to algebras of isomorphism types:

Theorem B.6. *Let $\mathfrak{A}$ be an algebra which satisfies* B.1(i), *and let $\tau(\mathfrak{A}) = \alpha$. If β_i for every $i \varepsilon I$ and γ_j for every $j \varepsilon J$ are types such that*

$$\prod_{i \in I} \beta_i \; \varepsilon \; \Phi(\alpha) \quad \textit{and} \quad \prod_{i \in I} \beta_i = \prod_{j \in J} \gamma_j \,,$$

then there is a double sequence of types $\delta_{i,j} \; \varepsilon \; \Phi(\alpha)$ *such that*

$$\beta_i = \prod_{j \in J} \delta_{i,j} \quad \textit{for every} \quad i \; \varepsilon \; I, \quad \textit{and} \quad \gamma_j = \prod_{i \in I} \delta_{i,j} \quad \textit{for every} \quad j \; \varepsilon \; J.$$

Similarly for strong and weak products. Hence, in particular, the algebra $\langle \Phi(\alpha), \times \rangle$ *has the refinement property.*

PROOF: by A.4, A.15, A.16, B.3(ii), and B.5 (cf. also 13.18).

As was pointed out in §12 (cf. the remarks which follow 12.14), the general refinement property formulated in B.6 has various consequences concerning the representation of a type as a product of indecomposable factors. These consequences will be stated explicitly in B.10(v),(vi)—in application to an especially important class of algebras which satisfy B.1(i).

THEOREM B.7. *If* $\mathfrak{A}$ *is an algebra satisfying* B.1(i) *and if* $\tau(\mathfrak{A}) = \alpha$, *then the algebra* $\langle \Phi(\alpha), \times, \prod \rangle$ *is isomorphic with the coset algebra* $\langle \mathsf{F}(\mathfrak{A}), \times, \prod \rangle / \cong$ *(where* $\cong$ *is the isomorphism relation between factors of* $\mathfrak{A}$*).*

PROOF: analogous to that of 12.15; we apply 6.3, 6.4, A.4, A.12, A.17, and B.3.

For weak products an analogous result—without any restrictions regarding $\mathfrak{A}$—has been stated in A.18(ii). Both results remain valid if '$\prod_w$' and '$\prod$' are understood to denote operations on arbitrary systems (and not only on infinite sequences); and in B.7 we can in this case replace '$\prod$' by '$\prod_s$'.

In Theorems B.8–B.10 we discuss a rather comprehensive class of algebras to which the results stated in B.1–B.7 can be applied. This class is constituted by those algebras $\mathfrak{A}$ in which Theorem 1.33 holds; such algebras will occasionally be referred to as ALGEBRAS WITH A STRONG ZERO ELEMENT. In particular, all partially ordered groupoids and generalized groupoids—thus, all C.A.'s, G.C.A.'s, partially ordered semigroups, and lattices—are algebras with a strong zero.

THEOREM B.8. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be an algebra in which, for all* $a, b \; \varepsilon \; A$, $a + b = 0$ *implies* $a = b = 0$. *We then have for every factor* B *of* $\mathfrak{A}$*:*

(i) *if* $a_1, a_2 \varepsilon A$ *and* $a_1 + a_2 \varepsilon B$, *then* $a_1, a_2 \varepsilon B$;
(ii) *if* $a \varepsilon A$, $b_i \varepsilon B$ *for every* $i \varepsilon I$, *and*

$$a = \bigcup_{i \varepsilon I} b_i$$

(*or, more generally, if* a *is an element such that* $b_i \leqq a$ *for every* $i \varepsilon I$, *and such that* $b_i \leqq x$ *for every* $i \varepsilon I$ *implies* $a \leqq x$), *then* $a \varepsilon B$;
(iii) *if* $a \varepsilon A$, $b_i \varepsilon B$ *for* $i < n \leqq \infty$, *and*

$$a = \sum_{i<n} b_i,$$

then $a \varepsilon B$.

Proof: By A.16 we have for some subalgebra C of $\mathfrak{A}$

$$A = B \times C.$$

Hence, in the case (i), we obtain by A.11(i)

(1) $a_1 = b_1 + c_1$ and $a_2 = b_2 + c_2$ where $b_1, b_2 \varepsilon B$ and $c_1, c_2 \varepsilon C$.

Consequently,

(2) $$(a_1 + a_2) + 0 = (b_1 + c_1) + (b_2 + c_2).$$

Since $a_1 + a_2 \varepsilon B$ and $0 \varepsilon C$, an application of A.11(iii) to (2) gives

$$0 = c_1 + c_2.$$

Hence, by hypothesis,

$$c_1 = c_2 = 0,$$

and therefore, by (1), $a_1, a_2 \varepsilon B$.

In the case (ii) we conclude from Definition 1.5, 3.2, and 13.2 that there are elements $a_i \varepsilon A$ such that

(3) $$a = b_i + a_i \quad \text{for} \quad i \varepsilon I.$$

By A.11(i) we obtain

(4) $$a = b + c \quad \text{with} \quad b \varepsilon B \quad \text{and} \quad c \varepsilon C$$

as well as

(5) $$a_i = b_i' + c_i \quad \text{with} \quad b_i' \varepsilon B \quad \text{and} \quad c_i \varepsilon C \quad \text{for every} \quad i \varepsilon I.$$

(3)–(5) give

$$b + c = (b_i + 0) + (b'_i + c_i);$$

hence, by A.11(iii),

$$b = b_i + b'_i$$

and therefore

$$b_i \leqq b \quad \text{for every} \quad i \in I.$$

By the hypothesis of (ii), this implies

$$a \leqq b,$$

i.e.,

$$b = a + a' \quad \text{for some} \quad a' \in A;$$

and since b is in B by (4), we conclude by applying part (i) of our theorem that a is also in B. Part (ii) has thus been established; part (iii) follows by 13.2 and A.20.

By B.8(i),(iii), a factor B of an algebra $\mathfrak{A}$ with a strong zero is an ideal in $\mathfrak{A}$ in the sense of 9.1. It is easily seen that, whenever the direct product of two (disjoint) ideals B and C in such an algebra $\mathfrak{A}$ exists, it coincides with the ideal sum of B and C in the sense of 10.1. In C.A.'s and finitely closed G.C.A.'s the converse also holds: the ideal sum of two disjoint ideals B and C is always the direct product of these ideals; and hence the factor algebra $\mathfrak{F}(\mathfrak{A})$ is a subalgebra of the disjunctive ideal algebra $\mathfrak{J}(\mathfrak{A})$ of §10. This explains why we could make such an extensive use of the properties of ideals in studying isomorphism types of G.C.A.'s in §12.

In algebras with a strong zero a direct product and a strong direct product of subalgebras can be characterized in terms of arithmetical properties of elements. (Regarding weak products cf. the remark which follows A.14.) Such a characterization is, however, rather involved. It becomes simpler if we restrict ourselves to a more special class of algebras—in fact, to those which are partially ordered in the sense of 13.10. We can then apply essentially the same method as in A.11—with sums replaced by least upper bounds. This fact will be used implicitly in the proofs of C.6 and C.7 in the next section. An implicit characterization of direct products in G.C.A.'s was given in 12.11.

THEOREM B.9. *Under the assumption of* B.8, $\mathfrak{F}(\mathfrak{A})$ *is a disjunctive Boolean algebra.*

PROOF: Let B_1, B_2, C_1, and C_2 be any factors of $\mathfrak{A}$ such that $B_1 \times B_2$ is also a factor and

$$(1) \qquad B_1 \times B_2 = C_1 \times C_2 .$$

Then, by A.10 and A.12, the product $(B_1 \cap C_1) \times (B_1 \cap C_2)$ exists, and, by A.3, we have

$$(2) \qquad (B_1 \cap C_1) \times (B_1 \cap C_2) \leqq B_1 .$$

Given any element $b \, \varepsilon \, B_1$, we obtain by (1), A.3, and A.11(i),

$$b = c_1 + c_2 \quad \text{for some} \quad c_1 \, \varepsilon \, C_1 \quad \text{and} \quad c_2 \, \varepsilon \, C_2 .$$

Hence, by B.8(i), c_1 and c_2 are in B_1. Therefore

$$c_1 \, \varepsilon \, B_1 \cap C_1 , \quad c_2 \, \varepsilon \, B_1 \cap C_2 ,$$

and consequently, by A.11(ii),

$$b = c_1 + c_2 \, \varepsilon \, (B_1 \cap C_1) \times (B_1 \cap C_2).$$

This shows that

$$B_1 \leqq (B_1 \cap C_1) \times (B_1 \cap C_2);$$

so that finally, in view of (2),

$$B_1 = (B_1 \cap C_1) \times (B_1 \cap C_2).$$

In a similar way we derive from (1) analogous formulas for B_2, C_1, and C_2. The sets $B_1 \cap C_1$, $B_1 \cap C_2$, etc. are factors of $\mathfrak{A}$ by A.19. Thus, according to 13.18, the factor algebra $\mathfrak{F}(\mathfrak{A})$ has the refinement property. The conclusion follows by B.1.

By B.9, Theorem B.6 applies to factors of a type $\alpha = \tau(\mathfrak{A})$ where $\mathfrak{A}$ is an arbitrary algebra with a strong zero. This result can be slightly improved:

THEOREM B.10. *Let* T *be the class of all types of algebras* $\mathfrak{A}$ *which satisfy the hypothesis of* B.8.

(i) *If* $\alpha \, \varepsilon \, \mathrm{T}$ *and* $\beta \, \varepsilon \, \Phi(\alpha)$, *then* $\beta \, \varepsilon \, \mathrm{T}$.

(ii) *If* $\alpha_i \, \varepsilon \, \mathrm{T}$ *for every* $i \, \varepsilon \, I$, *then* $\prod_{i \varepsilon I} \alpha_i \, \varepsilon \, \mathrm{T}$.

(iii) *The algebra* $\langle \mathrm{T}, \times \rangle$ *is a groupoid with the refinement property (and in which* 1.33 *holds).*

(iv) *If* $\alpha_i \varepsilon \mathrm{T}$ *for every* $i \varepsilon I$, $\beta_j \varepsilon \mathrm{T}$ *for every* $j \varepsilon J$, *and*

$$\prod_{i \varepsilon I} \alpha_i = \prod_{j \varepsilon J} \beta_j ,$$

then there are types $\gamma_{i,j} \varepsilon \mathrm{T}$ *such that*

$$\alpha_i = \prod_{j \varepsilon J} \gamma_{i,j} \quad \textit{for every} \quad i \varepsilon I, \quad \textit{and} \quad \beta_j = \prod_{i \varepsilon I} \gamma_{i,j} \quad \textit{for every} \quad j \varepsilon J.$$

(v) *Every type* $\alpha \varepsilon \mathrm{T}$ *has, apart from order, at most one representation*

$$\alpha = \prod_{i \varepsilon I} \alpha_i$$

where all types α_i *with* $i \varepsilon I$ *are indecomposable.*

(vi) *If a type* $\alpha \varepsilon \mathrm{T}$ *has a representation described in* (v), *and* $\beta \varepsilon \Phi(\alpha)$, *then* β *has also a similar representation*

$$\beta = \prod_{i \varepsilon J} \alpha_i$$

where J *is a subset of* I.

Conditions (ii) *and* (iv)–(vi) *apply also to strong and weak products.*

Proof: (i) and (ii) can be verified directly. Hence we obtain (iii) and (iv) by 13.18, A.1, B.6, and B.9; in applying B.6 we take for α the product of all types involved. To derive (v) and (vi) from (iv), we use A.2 and we argue analogously as in the proofs of 4.44 and 4.45.

Conclusions B.10(i),(ii) imply that the class T is an ideal, in the sense of 9.1, in the algebra of all isomorphism types under the operations $\times$ and $\prod$. B.10(iv) is a far-reaching generalization of 12.14.

Theorems B.9 and B.10 can be extended to a much wider class of algebras characterized in a rather simple way in terms of arithmetical properties of their elements. Let us agree to call an algebra $\mathfrak{A} = \langle A, + \rangle$ centerless if 0 is the only element $z \varepsilon A$ such that

(i) $z + z' = 0$ for some $z' \varepsilon A$;

(ii) whenever x, y, and $x + y$ are in A, $(x + y) + z$ is also in A, and we have

$$(x + y) + z = x + (y + z) = (x + z) + y.$$

The class of centerless algebras clearly contains all algebras with a strong zero; but it contains also many other algebras, e.g., all multiple-free algebras and all so-called centerless groups. It now turns out that every centerless algebra satisfies the conditions of B.1.[5] The proof will not be given here; the extension of B.9 thus obtained has but little significance for the study of algebras in which the commutative and associative laws hold.

[5] Only special cases of this result can be found in the literature. Cf. Fitting [1], p. 392, and Golowin [1], p. 424, for centerless groups, and Birkhoff [4], p. 611, for lattices; in these papers not condition B.1(i) but the equivalent condition B.1(ii) and the general refinement property formulated in B.3 are discussed.

C. ALGEBRAS WHOSE FACTOR ALGEBRAS ARE BOOLEAN AND COUNTABLY COMPLETE

We now turn to those algebras $\mathfrak{A}$ which have countably complete factor algebras.

THEOREM C.1. *Let $\mathfrak{A}$ be an algebra for which $\mathfrak{F}(\mathfrak{A})$ is a countably complete disjunctive Boolean algebra, and let $\tau(\mathfrak{A}) = \alpha$. Then $\langle \Phi(\alpha), \times \rangle$ is an R.A.*

PROOF: By 15.24, $\mathfrak{F}(\mathfrak{A})$ is a G.C.A. Hence, by arguing as in the proof of 11.35 and 11.36(ii), we show that $\mathfrak{F}(\mathfrak{A})/\cong$ is an R.A. By now applying A.18(ii), we arrive at the conclusion.

THEOREM C.2. *Let $\mathfrak{A} = \langle A, + \rangle$ be a G.C.A. such that*

(i) $a + b$ is in A whenever a and b are in A and $a \cap b = 0$.

Then $\mathfrak{F}(\mathfrak{A})$ is a countably complete disjunctive Boolean algebra.

PROOF: $\mathfrak{F}(\mathfrak{A})$ is a disjunctive Boolean algebra by B.9. Hence, to prove that it is countably complete, it suffices to establish the existence of the least upper bound of an arbitrary infinite sequence of pairwise disjoint factors $B_0, B_1, \cdots, B_i, \cdots$. Since, by A.22(v), A is the largest factor of $\mathfrak{A}$, i.e., the largest element of $\mathfrak{F}(\mathfrak{A})$, and the operation $\times$ plays the role of disjoint addition in $\mathfrak{F}(\mathfrak{A})$, we can construct, by using elementary properties of Boolean algebras, an infinite sequence of factors $A_0, A_1, \cdots, A_i, \cdots$ such that

(1) $A_0 = A$, and $A_n = B_n \times A_{n+1}$ for $n = 0, 1, 2, \cdots$.

Let B be the set of elements b of the form

(2) $$b = \sum_{i<\infty} b_i \quad \text{where} \quad b_i \,\varepsilon\, B_i \quad \text{for} \quad i < \infty;$$

and let C be the intersection of all factors $A_0, A_1, \cdots, A_i, \cdots$. We are going to show that

(3) $$A = B \times C.$$

By applying A.11(i) to (1) infinitely many times, we construct, for any given $a \,\varepsilon\, A$, two infinite sequences of elements $a_i \,\varepsilon\, A_i$ and $b_i \,\varepsilon\, B_i$ with

(4) $a_0 = a$, and $a_n = b_n + a_{n+1}$ for $n = 0, 1, 2, \cdots$.

By 5.1.V we conclude that there is an element c such that

(5) $$a_n = c + \sum_{i<\infty} b_{n+1} \quad \text{for} \quad i < \infty.$$

Let b be the element defined by (2); it clearly belongs to B. On the other hand, by (5) and B.8(i), c belongs to each of the factors $A_0, A_1, \cdots, A_i, \cdots$ and hence also to their intersection C. By (2), (4), and (5) we have

$$a = b + c.$$

Thus, condition B.11(i) is satisfied for $B = A$, $B_0 = B$, $B_1 = C$, and $n = 2$. Given now any elements $b \varepsilon B$ and $c \varepsilon C$, we put b in the form (2). We have

$$b_n \varepsilon B_n \quad \text{and} \quad c \varepsilon A_{n+1} \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

Therefore, by (1) and A.21(iv),

$$b_n \cap c = 0;$$

and consequently, by 3.12 and (2),

$$b \cap c = 0.$$

Hence, by hypothesis, $b + c$ is in A, and condition A.11(ii) holds. Finally, consider any elements $b, b', b'' \varepsilon B$ and $c, c', c'' \varepsilon C$. If

(6) $$b = b' + b'' \quad \text{and} \quad c = c' + c'',$$

we obviously have

(7) $$b + c = (b' + c') + (b'' + c'').$$

If, conversely, (7) holds, we obtain

(8) $$b + c = (b' + b'') + (c' + c'').$$

Since b' and b'' are of the form (2), the same applies to $b' + b''$ by 5.1.II, and therefore $b' + b''$ is in B. Furthermore, since c' and c'' are in A_n, their sum $c' + c''$ is also in A_n for every $n < \infty$ (cf. A.20), and hence it is in C. We show now, as before, that

(9) $$b \cap (c' + c'') = 0 = (b' + b'') \cap c;$$

and from (8) and (9) we easily obtain (6) by applying 3.3. Thus, condition A.11(iii) is also satisfied; and formula (3) has been established. According to A.16, this shows that B is a factor of $\mathfrak{A}$.

Every element b in B_n has clearly a representation (2) in which $b_i = 0$ for $i \neq n$. Hence

$$B_n \leqq B \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

Now let X be any factor in $\mathfrak{A}$ such that

$$B_n \leqq X \quad \text{for} \quad n = 0, 1, 2, \cdots;$$

and let a be any element in B, i.e., an element of the form (2). Since all the elements $b_0, b_1, \cdots, b_i, \cdots$ are in X, we conclude by B.8(iii) that b is also in X. Thus,

$$B \leqq X.$$

Hence, according to 3.2 and in view of A.24, B is the least upper bound of the factors $B_0, B_1, \cdots, B_i, \cdots$ in the algebra $\mathfrak{F}(\mathfrak{A})$. This completes the proof.

THEOREM C.3. *Let $\mathfrak{A} = \langle A, + \rangle$ be a partially ordered semigroup which satisfies the following condition:*

(i) *if $a_0, a_1, \cdots, a_i, \cdots, c$ are in A, $a_i \leqq c$ for every $i < \infty$, and $a_i \cup a_j = a_i + a_j$ for $i < j < \infty$, then $\sum_{i<\infty} a_i$ exists and is in A. Under these assumptions $\mathfrak{F}(\mathfrak{A})$ is a countably complete disjunctive Boolean algebra.*

PROOF: As in the proof of C.2 we have to show that every infinite sequence of pairwise disjoint factors $B_0, B_1, \cdots, B_i, \cdots$ has a least upper bound in the algebra $\mathfrak{F}(\mathfrak{A})$. By A.22(v) there are factors $C_0, C_1, \cdots, C_i, \cdots$ such that

$$(1) \qquad A = B_i \times C_i \quad \text{for} \quad i < \infty;$$

moreover, $\mathfrak{F}(\mathfrak{A})$ being a disjunctive Boolean algebra by 13.11 and B.9, we conclude that

$$(2) \qquad B_i \times B_j \quad \text{is a factor for} \quad i < j < \infty.$$

Let B be the set of all elements b of the form

$$(3) \qquad b = \sum_{i<\infty} b_i \quad \text{where} \quad b_i \in B_i \quad \text{for} \quad i = 0, 1, 2, \cdots;$$

and let C be the intersection of the factors $C_0, C_1, \cdots, C_i, \cdots$. For any given element $a \in A$ we obtain, by (1) and A.11(i), two sequences of elements b_i and c_i such that

$$(4) \quad a = b_i + c_i, \quad b_i \in B_i, \quad \text{and} \quad c_i \in C_i \quad \text{for} \quad i = 0, 1, 2, \cdots.$$

We have, by (2), (4), and A.21(iv),

$$b_i \cup b_j = b_i + b_j \quad \text{for} \quad i < j < \infty. \tag{5}$$

By hypothesis, (4) and (5) imply the existence of the element $b \in B$ defined by formula (3); and we have by 13.17

$$b = \bigcup_{i<\infty} b_i. \tag{6}$$

Hence, by (4),

$$b \leqq a,$$

and therefore, for some $c \in A$,

$$a = b + c. \tag{7}$$

By 13.2, 13.16(i), and an easy induction, we obtain from (2), (4), and (6)

$$b_n + c_n = b_n + \sum_{i<n} b_i + \sum_{i<\infty} b_{n+i} + c,$$

and consequently

$$c_n = \sum_{i<n} b_i + \sum_{i<\infty} b_{n+i} + c.$$

Therefore, by (4) and B.8(i), c is in C_n for $n = 0, 1, 2, \cdots$, and hence also in C; while the element b defined by (2) is clearly in B. Thus, in view of (7), every element $a \in A$ has a representation required in A.11(i) (for $B = A$, $B_0 = B$, $B_1 = C$, and $n = 2$).

Condition A.11(ii) is obviously satisfied. To obtain A.11(iii), consider arbitrary elements $b, b', b'' \in B$ and $c, c', c'' \in C$. If

$$b = b' + b'' \quad \text{and} \quad c = c' + c'', \tag{8}$$

we at once obtain

$$b + c = (b' + c') + (b'' + c''). \tag{9}$$

If, conversely, (9) holds, we have

$$b + c = (b' + b'') + (c' + c''). \tag{10}$$

We show, as in the preceding proof, that $b' + b''$ is in B and $c' + c''$ is in C; we apply here 13.16(ii) instead of 5.1.II. The element b has a representation (2); we show, as before, that it also satisfies (6). Since b_i is in B_i and $c' + c''$ is in C, we have, by (1) and A.21(iv),

$$b_i \cup (c' + c'') = b_i + (c' + c'').$$

Consequently, by (6) and 13.14(i),

(11) $b \cup (c' + c'') = (\bigcup_{i<\infty} b_i) \cup (c' + c'') = \bigcup_{i<\infty} [b_i \cup (c' + c'')]$

$$= \bigcup_{i<\infty} [b_i + (c' + c'')] = (\bigcup_{i<\infty} b_i) + (c' + c'') = b + (c' + c'').$$

Similarly we obtain

(12) $$(b' + b'') \cup c = (b' + b'') + c.$$

In view of (10) we have

$$b \leqq b + c \quad \text{and} \quad c' + c'' \leqq b + c;$$

hence, by (11) and 3.2,

$$b + (c' + c'') \leqq b + c = (b' + b'') + (c' + c'');$$

and furthermore, by the cancellation law,

$$b \leqq b' + b'' \quad \text{and} \quad c' + c'' \leqq c.$$

The inequalities in the opposite direction can be derived in an analogous way from (12), so that finally we arrive at equations (8). Thus, condition A.11(iii) is satisfied. Hence, A is the direct product of B and C, and B is a factor of $\mathfrak{A}$. To complete the proof we have to show that B is the least upper bound of $B_0, B_1, \cdots, B_i, \cdots$; this can be done by arguing exactly as in the proof of C.2.

The class of semigroups involved in C.3 contains, in particular, all semigroups discussed in 13.27 and 13.28—thus, all semigroups which are G.C.A.'s.

We could replace in C.3(i) the formula

$$a_i \cup a_j = a_i + a_j$$

by

$$a_i \cap a_j = 0;$$

this, however, would weaken our result (cf. the remarks which follow 13.17). The same applies to Theorem C.7 below. On the other hand, we could improve the result obtained in C.3 by introducing, in the hypothesis of C.3(i), a second sequence of elements b_i subjected to the condition

$$c = a_i + b_i = a_i \cup b_i \text{ (or } c = a_i + b_i \text{ and } a_i \cap b_i = 0) \text{ for every } i < \infty.$$

THEOREM C.4. *Let* T *be the class of types of all G.C.A.'s* $\mathfrak{A}$ *which satisfy* C.2(i), *or of all semigroups* $\mathfrak{A}$ *which satisfy* C.3(i). *Then*

(i) *conclusions* B.10(i),(ii) *hold;*

(ii) *the algebra* $\langle T, \times \rangle$ *is a finitely closed R.A.*

PROOF: by A.1 and C.1–C.3 (cf. the proof of B.10).

C.1 and C.4 have some rather interesting consequences for the isomorphism types involved (cf. the remarks after 12.21); these consequences are not, however, very numerous. We obtain new results in the arithmetic of types by subjecting the algebras $\mathfrak{A}$ to further restrictions. In the algebras discussed in C.1–C.4 every sequence of disjoint factors has a least upper bound, but nothing more precise is known about the relation between the terms of a sequence of factors and their bound; it may happen, for instance, that two sequences of pairwise isomorphic factors have least upper bounds which are not isomorphic. If in an algebra $\mathfrak{A}$—for which $\mathfrak{F}(\mathfrak{A})$ is a countably complete disjunctive Boolean algebra—such a possibility is excluded, then the correlated algebra of types $\langle \Phi(\alpha), \times \rangle$ proves to be a G.C.A., and not only an R.A. This will be the case in all the algebras $\mathfrak{A}$ we are going to discuss in the remaining part of this section.

THEOREM C.5. *Let* $\mathfrak{A}$ *be an algebra which satisfies* B.1(i) *as well as the following condition:*

(i) *if* $B_0, B_1, \cdots, B_i, \cdots$ *are factors of* $\mathfrak{A}$, *and if* $B_i \cap B_j = \{0\}$ *for* $i < j < \infty$, *then* $\prod_{i<\infty} B_i$ *exists and is a factor of* $\mathfrak{A}$.
Let, moreover, $\tau(\mathfrak{A}) = \alpha$. *Then* $\langle \Phi(\alpha), \times, \prod \rangle$*—as well as* $\langle \Phi(\alpha), \times \rangle$*—is a G.C.A.*

PROOF: From the hypothesis we easily obtain by means of A.22(iii), B.2, and B.3:

$$(1) \qquad \prod_{i<\infty} B_i = \mathsf{U}_f\, B_i = \mathsf{U}_f \prod_{i<n} B_i$$

with $i<\infty$ under the first U_f and $n<\infty$ under the second, for every sequence of factors $B_0, B_1, \cdots, B_i, \cdots$ which are pairwise disjoint in the factor algebra $\mathfrak{F}(\mathfrak{A})$. Hence we conclude that $\mathfrak{F}(\mathfrak{A})$ is a countably complete disjunctive Boolean algebra, and consequently, by 15.24, it is a G.C.A. Moreover, we see from (1), 13.1, and 13.3 that $\langle \mathsf{F}(\mathfrak{A}), \times, \prod \rangle$ is a G.C.A. Hence, by 6.10 and A.18(i), the coset algebra $\langle \mathsf{F}(\mathfrak{A}), \times, \prod \rangle/\cong$ is also a G.C.A. The conclusion now follows at once by B.7.

THEOREM C.6. *Let $\mathfrak{A} = \langle A, + \rangle$ be a G.C.A. with the following property:*

(i) $\sum_{i<\infty} a_i$ *is in* A *whenever* $a_0, a_1, \cdots, a_i, \cdots$ *are in* A *and* $a_i \cap a_j = 0$ *for* $i < j < \infty$.
Then $\mathfrak{A}$ *satisfies* B.1(i) *and* C.5(i).

PROOF: By B.9, $\mathfrak{A}$ satisfies B.1(i). Given a sequence of factors $B_0, B_1, \cdots, B_i, \cdots$ with

(1) $$B_i \cap B_j = \{0\} \quad \text{for} \quad i < j < \infty,$$

let B be the set of all elements b of the form

(2) $$b = \sum_{i<\infty} b_i \quad \text{where} \quad b_i \, \varepsilon \, B_i \quad \text{for every} \quad i < \infty.$$

By (1), A.22(iii), and A.26(i), the factors B_i are pairwise disjoint in the algebra $\mathfrak{F}(\mathfrak{A})$, and we have

(3) $$A_i \times A_j \, \varepsilon \, \mathsf{F}(\mathfrak{A}) \quad \text{for} \quad i < j < \infty.$$

We now repeat the proof of C.2 and show that the set B is itself a factor. Furthermore, it is easily seen that every element b in B has only one representation (2). For, given another representation of this kind:

(4) $$b = \sum_{i<\infty} b'_i \quad \text{where} \quad b'_i \, \varepsilon \, B_i \quad \text{for every} \quad i < \infty,$$

we obtain, by (3) and A.21(iv),

$$b_i \cap b'_j = 0 \quad \text{for} \quad i < \infty, \quad j < \infty, \quad \text{and} \quad i \neq j;$$

hence and from (2) and (4) we easily conclude by 3.11 that

$$b_i = b'_i \quad \text{for} \quad i = 0, 1, 2, \cdots.$$

Thus, we can correlate with every element $b \, \varepsilon \, B$ a uniquely determined sequence of elements b_i satisfying (2). Conversely, given a sequence of elements $b_i \, \varepsilon \, B_i$ for $i = 0, 1, 2, \cdots$, we have, again by (3) and A.21(iv),

$$b_i \cap b_j = 0 \quad \text{for} \quad i < j < \infty;$$

and hence we conclude by hypothesis that an element b defined by (2) exists, and therefore belongs to B. Consider, finally, any three elements $b, b', b'' \, \varepsilon \, B$ and three correlated sequences b_i, b'_i, b''_i. If

(5) $$b_i = b'_i + b''_i \quad \text{for} \quad i < \infty,$$

we have by 5.1.II

(6) $$b = b' + b''.$$

If, conversely, (6) holds, we easily obtain—e.g., with the help of 5.22—

$$b = \sum_{i<\infty} b_i = \sum_{i<\infty} (b'_i + b''_i)$$

where, by A.20, the elements $b'_i + b''_i$ are in B_i for every $i < \infty$; and hence, in view of the unicity of representation (2), condition (5) must hold. Thus we see that our correlation establishes an isomorphism between the algebras $\langle B, + \rangle$ and $\prod_{i<\infty} \langle B_i, + \rangle$. Moreover, for every $j < \infty$, the set B_j consists of those and only those elements $b \in B$ for which in the correlated sequence we have

$$b_j = b, \quad \text{and} \quad b_i = 0 \quad \text{for} \quad i \neq j;$$

for

$$b = \sum_{i<\infty} b_i \in B_j$$

implies

$$b_i \leqq b \quad \text{and} \quad b_i \cap b = 0 \quad \text{for} \quad i < \infty, i \neq j.$$

Hence, by A.3 and B.3(ii),

$$B = \prod_{i<\infty} B_i.$$

Condition C.5(i) is thus satisfied, and the proof is complete.

THEOREM C.7. *Let $\mathfrak{A} = \langle A, + \rangle$ be a partially ordered semigroup with the following property:*

(i) *$\sum_{i<\infty} a_i$ is in A whenever $a_0, a_1, \cdots, a_i, \cdots$ are in A and $a_i \cup a_j = a_i + a_j$ for $i < j < \infty$.*

Then $\mathfrak{A}$ satisfies B.1(i) *and* C.5(i).

PROOF: The reasoning is analogous to that in the preceding proof; certain changes are required, however. To show that B is a factor, we repeat the proof of C.3 (and not of C.2). To show that every element $b \in B$ has only one representation (2), we consider another such representation (4), and we put, for a given $m < \infty$,

$$a_m = b'_m, \quad \text{and} \quad a_i = b_i \quad \text{for} \quad i < \infty, \quad i \neq m.$$

We have, by (3) and A.21(iv),

$$a_i \cup a_j = a_i + a_j \quad \text{for} \quad i < j < \infty;$$

hence, by 13.17,

$$\sum_{i<\infty} a_i = \bigcup_{i<\infty} a_i .$$

Moreover, as is easily seen from (2), (4), and 13.2,

$$a_j \leqq b = \sum_{i<\infty} b_i \quad \text{for} \quad j = 0, 1, 2, \cdots .$$

Therefore

$$\sum_{i<\infty} a_i = \bigcup_{i<\infty} a_i \leqq \sum_{i<\infty} b_i .$$

Hence, by means of 13.16(i) and an easy induction,

$$\sum_{i<m} a_i + a_m + \sum_{i<\infty} a_{m+1+i} \leqq \sum_{i<m} b_i + b_m + \sum_{i<\infty} b_{m+1+i} ;$$

consequently

$$\sum_{i<m} b_i + b'_m + \sum_{i<\infty} b_{m+1+i} \leqq \sum_{i<m} b_i + b_m + \sum_{i<\infty} b_{m+1+i} ;$$

and finally, by applying the cancellation law,

$$b'_m \leqq b_m .$$

Similarly, we obtain the inequality in the opposite direction, so that

$$b_m = b'_m \quad \text{for every} \quad m < \infty .$$

The only difference in the remaining part of the proof consists in the fact that 13.16(ii) is applied instead of 5.11 and 5.22.

THEOREM C.8. *Let* T *be the class of types of all G.C.A.'s* $\mathfrak{A}$ *which satisfy* C.6(i), *or of all partially ordered semigroups* $\mathfrak{A}$ *which satisfy* C.7(i). *Then*

(i) *conclusion* B.10(i) *holds, and also conclusion* B.10(ii) *for plain and strong products;*

(ii) *the algebra* $\langle T, \times, \prod \rangle$*—as well as* $\langle T, \times \rangle$*—is a C.A.*

PROOF: By A.1 and C.5–C.7 (cf. the proof of B.10).

It would be interesting to find a class of algebras characterized in a simple and 'natural' way, which would comprehend all algebras of Theorem C.6 and C.7 (or C.2 and C.3), and to which the common

conclusion of these theorems, as well as the conclusion of C.8 (or C.4), would apply. This is not, however, quite easy since the algebras of C.6 differ considerably from those of C.7. An essential part in the proof of C.6 is played by the remainder postulate and by some consequences of the refinement postulate, like 13.11 or 13.12; whereas analogous conclusions in the proof of C.7 have been obtained by means of the cancellation law.

Theorems C.4 and C.8 contain generalizations of Theorems 12.17, 12.21, and 13.30 which concern types of C.A.'s, finitely closed G.C.A.'s, and certain special semigroups. Moreover, C.8 gives an essentially new result which applies to types of a rather comprehensive class of semigroups. By means of similar methods we could obtain generalizations of Theorems 15.27 and 15.28 applying to types of certain lattices and Boolean algebras.

Under the assumption that the factor algebra $\mathfrak{F}(\mathfrak{A})$ is a disjunctive Boolean algebra, condition C.5(i) clearly implies that this factor algebra is countably complete (cf. the proof of C.5). It will be seen from Theorem C.10 that an analogous condition for weak products carries with it much stronger consequences. In fact, the factor algebra proves then to be complete and what is called ATOMISTIC (i.e., every factor is a least upper bound of indecomposable factors). Hence, as is well known, it follows that $\mathfrak{F}(\mathfrak{A})$ is isomorphic with the family of all subsets of a set S under (disjunctive) set-theoretical addition; in fact, the set of all indecomposable factors of $\mathfrak{A}$ can be taken for S.[6] The condition in question has also some further interesting consequences both for the algebra $\mathfrak{A}$ and for the correlated algebra of types $\langle \Phi(\alpha), \times \rangle$.

To formulate Theorem C.10 conveniently, we introduce the notion of a SIMPLE ELEMENT:

DEFINITION C.9. *$\mathfrak{A} = \langle A, + \rangle$ being an algebra, an element $a \; \varepsilon \; A$ is called SIMPLE if $a \neq 0$ and if, for all X and Y such that $A = X \times Y$, we have $a \; \varepsilon \; X$ or $a \; \varepsilon \; Y$.*

THEOREM C.10. *For every algebra $\mathfrak{A} = \langle A, + \rangle$ which satisfies* B.1(i) *the following conditions are equivalent:*

(i) *if $B_0, B_1, \cdots, B_i, \cdots$ are factors of $\mathfrak{A}$ and if $B_i \cap B_j = \{0\}$ for $i < j < \infty$, then $\prod^w_{i<\infty} B_i$ exists and is a factor of $\mathfrak{A}$;*

[6] Cf. Tarski [10], p. 197.

(ii) *if all sets* B_i *with* $i \varepsilon I$ *are factors of* $\mathfrak{A}$ *and if* $B_i \cap B_j = \{0\}$ *for all* $i, j \varepsilon I$ *with* $i \neq j$, *then* $\prod_{i \varepsilon I}^{w} B_i$ *exists and is a factor of* $\mathfrak{A}$;

(iii) *every factor* B *of* $\mathfrak{A}$ *can be represented in the form*

$$B = \prod_{i \varepsilon I}^{w} B_i$$

where all the factors B_i *with* $i \varepsilon I$ *are indecomposable in the algebra* $\mathfrak{F}(\mathfrak{A})$;

(iv) *every element* a *in* A *can be represented in the form*

$$a = \sum_{i < n} a_i$$

where $n < \infty$ *and all the elements* a_i *with* $i < n$ *are simple;*

(v) *if* $a_0, a_1, \cdots, a_i, \cdots, b_0, b_1, \cdots, b_i, \cdots$ *are elements of* A, *and* $A_0, A_1, \cdots, A_i, \cdots, B_0, B_1, \cdots, B_i, \cdots$ *are factors of* $\mathfrak{A}$, *and if* $a_n \varepsilon A_n$, $b_n \varepsilon B_n$, $a_n = b_n + a_{n+1}$, *and* $A_n = B_n \times A_{n+1}$ *for every* $n < \infty$, *then there is an integer* $m < \infty$ *such that* $b_{m+i} = 0$ *for* $i = 0, 1, 2, \cdots$.

PROOF: (I) We want first to show that (i) implies (v). Consider, in fact, elements a_i, b_i, and factors A_i, B_i such that

$$(1) \quad a_n \varepsilon A_n, \ b_n \varepsilon B_n, \quad a_n = b_n + a_{n+1}, \quad \text{and} \quad A_n = B_n \times A_{n+1} \quad \text{for} \quad n < \infty.$$

Hence, by an easy induction, with the help of A.13(v) and A.22(ii),

$$(2) \quad a_n = \sum_{i<p} b_{n+1} + a_{n+p} \quad \text{and} \quad A_n = \prod_{i<p} B_{n+i} \times A_{n+p} \quad \text{for} \quad n < \infty \quad \text{and} \quad p < \infty.$$

The second of these formulas implies, e.g., by A.7 and A.5, that

$$B_i \leqq_f A_0 \quad \text{for} \quad i = 0, 1, 2, \cdots.$$

Since, by (i), the product $\prod_{i \varepsilon I}^{w} B_i$ is a factor of $\mathfrak{A}$, we obtain hence, by A.28,

$$\prod_{i \varepsilon I}^{w} B_i \leqq_f A_0;$$

consequently there is a factor C such that

$$(3) \qquad A_0 = \prod_{i \varepsilon I}^{w} B_i \times C.$$

We have of course (by A.7 or A.28)

$$\prod_{i<p} B_i \leqq_f \prod_{i\varepsilon I}{}_w B_i \quad \text{for} \quad p < \infty;$$

hence, $\mathfrak{F}(\mathfrak{A})$ being by hypothesis a disjunctive Boolean algebra, we conclude from (2) and (3) that

$$(4) \qquad C \leqq_f A_p \quad \text{for} \quad p = 0, 1, 2, \cdots.$$

By (1), the element a_0 is in A_0. Together with (3), this implies by A.11(i) that, for some b and c,

$$(5) \qquad a_0 = b + c \quad \text{where} \quad b \,\varepsilon \prod_{i<\infty}{}_w B_i \quad \text{and} \quad c \,\varepsilon\, C.$$

By A.14(ii), there is an integer m such that b is in $\prod_{i<m} B_i$; while, by (4) and A.24, c belongs to all factors $A_0, A_1, \cdots, A_i, \cdots$, and in particular to A_m. Thus, by (1), (2), and (5), and in view of A.11(i), we have two decompositions of a_0:

$$a_0 = b + c = \sum_{i<m} b_i + a_m$$

where b and $\sum_{i<m} b_i$ are in $\prod_{i<m} B_i$, while c and a_m are in A_m. Since, by (2),

$$a_0 \,\varepsilon \prod_{i<m} B_i \times A_m,$$

we conclude by A.13(i) that

$$a_m = c.$$

Hence, by (1),

$$c = 0 + c = b_m + a_{m+1}; \quad c \,\varepsilon\, B_m \times A_{m+1}; \quad 0,\, b_m \,\varepsilon\, B_m;$$

$$c,\, a_{m+1} \,\varepsilon\, A_{m+1};$$

and, by applying A.13(i) again, we obtain

$$b_m = 0 \quad \text{and} \quad a_{m+1} = c.$$

By repeating this argument infinitely many times, we arrive at the conclusion of (v):

$$b_{m+i} = 0 \quad \text{for} \quad i = 0, 1, 2, \cdots.$$

(II) Next we want to derive (iv) from (v). Actually, we shall obtain a stronger conclusion:

(iv′) *if an element d belongs to a factor D, then there are simple elements $d_0, d_1, \cdots, d_i, \cdots$ and factors $D_0, D_1, \cdots, D_i, \cdots$ with $i < n < \infty$ such that*

(6) $\quad d = \sum_{i<n} d_i, \quad d_i \,\varepsilon\, D_i \quad \textit{for} \quad i < n, \quad \textit{and} \quad D = \prod_{i<n} D_i.$

In fact, let C be the set of all elements $c \,\varepsilon\, A$ which satisfy (iv′) for every factor D. Assume that a certain element a_0 is not in C; thus, for some factor A_0, a_0 belongs to A_0, but there are no simple elements d_i and factors D_i which satisfy (6) for $d = a_0$ and $D = A_0$. *A fortiori* a_0 is not simple itself. Hence, by C.9, there are sets X and Y such that

$$C = X \times Y,$$

and such that a_0 is neither in X nor in Y. Consequently, by B.4,

(7) $$A_0 = (A_0 \cap X) \times (A_0 \cap Y).$$

$A_0 \cap X$ and $A_0 \cap Y$ are factors by A.6, A.16, and B.2; and a_0 belongs to neither of them. By A.11(i) we conclude from (7) that, for some x and y,

(8) $\quad a_0 = x + y, \quad x \neq 0, \quad y \neq 0, \quad x \,\varepsilon\, A_0 \cap X, \quad \text{and} \quad y \,\varepsilon\, A_0 \cap Y.$

Assume that x and y are both in C. This implies that there are simple elements d'_i, d''_j and factors D'_i, D''_j with $i < n < \infty$ and $j < p < \infty$ such that

(9) $\quad x = \sum_{i<n} d'_i, \quad d'_i \,\varepsilon\, D'_i \text{ for } i < n, \quad \text{and} \quad A_0 \cap X = \prod_{i<n} D'_i,$

(10) $\quad y = \sum_{j<p} d''_j, \quad d''_j \,\varepsilon\, D''_j \text{ for } j < p, \quad \text{and} \quad A_0 \cap Y = \prod_{j<p} D''_j.$

By putting

$$d_j = d''_j \quad \text{and} \quad D_j = D''_j \quad \text{for} \quad j < p,$$
$$d_{p+i} = d'_i \quad \text{and} \quad D_{p+i} = D'_{p+i} \quad \text{for} \quad i < n,$$

we easily derive from (7)–(10), by means of A.7, A.11(iii), and A.12,

$$a_0 = \sum_{i<n+p} d_i, \quad d_i \,\varepsilon\, D_i \quad \text{for} \quad i < n + p, \quad \text{and} \quad A_0 = \prod_{i<n+p} D_i.$$

Thus the elements d_i (which are all simple) and the factors D_i satisfy formulas (6) for $d = a_0$ and $D = A_0$; however, this contradicts our previous assumption. Consequently, the elements x and y cannot both be in C; in view of (8) and A.13(iv), we can assume that y is not in C. We put

$$b_0 = x, \quad a_1 = y, \quad B_0 = A_0 \cap X, \quad \text{and} \quad A_1 = A_0 \cap Y.$$

By continuing this process indefinitely, we arrive at infinite sequences of elements a_i, b_i and factors A_i, B_i such that

$$a_n \varepsilon A_n, \quad b_n \varepsilon B_n, \quad b_n \neq 0, \quad a_n = b_n + a_{n+1}, \quad \text{and}$$

$$A_n = B_n \times A_{n+1}$$

for $n = 0, 1, 2, \cdots$. This, however, is inconsistent with (v). Thus if (v) holds, every element in A must belong to C; and therefore conditions (iv′) and (iv) are satisfied.

(III) In turn, we shall derive (iii) from (iv). Given a simple element a, let $G(a)$ be the intersection of all factors of $\mathfrak{A}$ which contain a. Consider any finite sequence of sets $G(a_0)$, $G(a_1)$, $\cdots$, $G(a_i)$, $\cdots$ with

$$(11) \qquad G(a_i) \neq G(a_j) \quad \text{for} \quad i < j < n.$$

We first show by induction with respect to n that there are factors D_0, D_1, $\cdots$, D_i, $\cdots$ which satisfy the formulas

$$(12) \qquad a_i \varepsilon D_i \text{ for } i < n \quad \text{and} \quad \prod_{i<n} D_i \varepsilon \mathsf{F}(\mathfrak{A}).$$

This is obvious by A.5, A.9(ii), and A.22(iii),(v) in case $n = 0, 1$. Assume now that $n = k + 1 \neq 1$ and that factors D_0', D_1', $\cdots$, D_i', $\cdots$ with the desired properties have been constructed for k elements a_0, a_1, $\cdots$, a_i, $\cdots$ with $i < k$. Suppose further that, for some $i < k$,

$$(13) \qquad a_i \varepsilon G(a_k).$$

Given a factor X which contains a_i, we have by A.22(v)

$$C = X \times Y$$

for a certain factor Y. Hence, by C.9, a_k is either in X or in Y. If a_k is in Y, then, by (13), a_i is in Y too; and hence, by A.9(i), $a_i = 0$, in contradiction to C.9. Thus, a_k is in X. Since X is an arbitrary factor containing b_i, we conclude that

$$(14) \qquad a_k \varepsilon G(a_i).$$

Formulas (13) and (14) clearly imply

$$G(a_i) = G(a_k);$$

this conclusion, however, contradicts (11). Consequently, none of the elements a_0, a_1, $\cdots$, a_i, $\cdots$ with $i < k$ is in $G(a_k)$. Hence,

by the definition of $G(a_k)$, there are factors $B_0, B_1, \cdots, B_i, \cdots$ such that a_k is in B_i but b_i is not in B_i for $i < k$. By A.22(v), we have for some factors C_i with $i < k$

(15) $$A = B_i \times C_i.$$

By C.9, a_i is in C_i for $i = 0, 1, \cdots, k - 1$. We now put

$$D_i = C_i \cap D'_i \quad \text{for} \quad i < k, \quad \text{and} \quad D_k = \bigcap_{i<k} B_i.$$

It is easily seen that the sets D_i thus defined are factors which satisfy (12) for $n = k + 1$; we make use here of the fact that $\mathfrak{F}(\mathfrak{A})$ is a disjunctive Boolean algebra, and we apply B.2 and (15). Thus, factors D_i with the required properties can be constructed for every number $n < \infty$.

(12) clearly implies

(16) $$G(a_i) \leqq D_i \quad \text{for} \quad i < n.$$

Hence, by A.10 and A.12, the product $\prod_{i<n} G(a_i)$ exists. Furthermore, we show that, for any elements b, b', and b'',

(17) $$b = b' + b'' \quad \text{and} \quad b', b'' \in \prod_{i<n} G(a_i) \quad \text{imply} \quad b \in \prod_{i<n} G(a_i).$$

In fact, by A.11(i), we have

$$b' = \sum_{i<n} b'_i \quad \text{and} \quad b'' = \sum_{i<n} b''_i \quad \text{where} \quad b'_i, b''_i \in G(a_i) \text{ for } i < n.$$

Hence, by (16) and A.20, the elements $b'_i + b''_i$ exist and are in D_i; and, in view of A.11(iii), we have

$$b = \sum_{i<n} (b'_i + b''_i).$$

From the definition of $G(a_i)$ we obtain further, with the help of A.20,

$$b'_i + b''_i \in G(a_i) \quad \text{for} \quad i < n;$$

and, by applying A.11(iii) again, we arrive at the conclusion of (17).

We have thus shown that all the finite products of $\prod_{i<n} G(a_i)$ of n distinct sets $G(a_0), G(a_1), \cdots, G(a_i), \cdots$ exist and that each of them satisfies (17). Given now any element a, we have by condition (iv) of our theorem

$$a = \sum_{i<m} d_i$$

where $m < \infty$ and the elements $d_0, d_1, \cdots, d_i, \cdots$ are simple. We show by induction with respect to m that a belongs to one of the finite products under discussion. By A.9(ii) and 13.2, this is obvious for $m = 0$. Assume that it holds for $m = k$. If now $m = k + 1$, we have

$$a = \sum_{i<k} d_i + d_k$$

where $\sum_{i<k} d_i$ belongs to a certain product $\prod_{i<n} G(a_i)$. In case $G(d_k)$ coincides with one of the sets $G(a_i)$ with $i < n$, d_k also belongs to this product by A.3, and the same applies to a by (17). Otherwise, we put

$$a_n = d_k ,$$

and we easily prove by A.8 and (17) that a belongs to $\prod_{i<n+1} G(a_i)$. Hence, A is the union of all products $\prod_{i<n} G(a_i)$. Therefore, by A.14 and A.15,

$$A = \prod_{X \varepsilon \mathsf{G}}^{w} X \tag{18}$$

where G is the family of sets $G(a)$ correlated with simple elements $a \varepsilon A$. Consequently, by A.19 and A.22(v), all these sets $G(a)$ are factors of $\mathfrak{A}$. Consider now any such factor $G(a)$. Thus

$$A = G(a) \times C \quad \text{for some factor} \quad C.$$

If

$$G(a) = X \times Y, \tag{19}$$

we have by A.22(i),(ii),(v)

$$A = (X \times Y) \times C = (X \times C) \times Y.$$

Therefore, by C.9, a is in $X \times C$ or in Y. In the first case we apply A.27 and we conclude that a is in

$$(X \times Y) \cap (X \times C) = X.$$

Hence, obviously $G(a)$ is included in X; consequently, by A.24,

$$X = G(a) \times Z \quad \text{for some factor } Z;$$

and therefore, by (19), A.7, and A.9(iii),

$$Y = \{0\}.$$

Similarly, if a is in Y, we obtain

$$X = \{0\}.$$

Thus, (19) implies that either X or Y coincides with $\{0\}$. However, the factor $G(a)$ itself is different from $\{0\}$ by C.9. Hence, according to 4.38, $G(a)$ is indecomposable in the algebra $\mathfrak{F}(\mathfrak{A})$. We have thus arrived in (18) at a representation of A as a product of indecomposable factors. Given now an arbitrary factor B, we obtain from (18) by B.2 and B.4

$$B = \prod_{X \varepsilon G}{}_w (B \cap_f X). \tag{20}$$

By A.7 and A.9, we can omit in (20) those factors $B \cap_f X$ which are equal to $\{0\}$; by 4.38, the remaining factors are indecomposable. Thus, B can also be represented as a product of indecomposable factors; and the derivation of (iii) is complete.

(IV) Now assume (iii) to hold and consider a representation of A as a product of indecomposable factors

$$A = \prod_{j \varepsilon J}{}_w A_j. \tag{21}$$

Given any factors B_i with $i \varepsilon I$ we easily conclude that each of them has a representation

$$B_i = \prod_{j \varepsilon J_i}{}_w A_j \tag{22}$$

where J_i is a subset of J; the proof is based essentially upon B.5 and is analogous to that of 4.44. If, moreover,

$$B_i \cap B_j = \{0\} \quad \text{for} \quad i,\ j \,\varepsilon\, I \quad \text{with} \quad i \neq j,$$

we easily see that the sets J_i must be mutually exclusive. Now let K be the union of all sets J_i with $i \varepsilon I$, and let L be the difference $J - K$. We obtain then from (21) and (22) by A.6, A.7, A.12, and A.15

$$A = \prod_{j \varepsilon K}{}_w A_j \times \prod_{j \varepsilon L}{}_w A_j \quad \text{and} \quad \prod_{j \varepsilon K}{}_w A_j = \prod_{i \varepsilon I}{}_w B_i .$$

Hence, by A.16, the product $\prod_{i \varepsilon I}{}_w B_i$ is a factor of $\mathfrak{A}$. Thus, (iii) implies (ii).

(V) Finally, (i) is but a particular case of (ii).

By (I)–(V), all the conditions (i)–(v) of our theorem are equivalent, and the proof is complete.

Several other conditions are known which are equivalent to those of Theorem C.10. For instance, in view of A.14(ii), condition C.10(i) can clearly be replaced by the following:

(i′) *if* $B_0, B_1, \cdots, B_\iota, \cdots$ *are factors of* $\mathfrak{A}$ *and if* $B_i \leqq B_j$ *for all* i *and* j *with* $i < j < \infty$, *then the union* $\bigcup_{\iota<\infty} B_i$ *is a factor of* $\mathfrak{A}$.

Similarly, C.10(ii) can be shown to be equivalent with the condition:

(ii′) *if all the sets* B_i *with* $i \varepsilon I$ *are factors of* $\mathfrak{A}$ *and if the family of these sets is simply ordered by the relation* $\leqq$, *then the union* $\bigcup_{i \varepsilon I} B_\iota$ *is a factor of* $\mathfrak{A}$.

C.10(iii) can be replaced by a weaker condition by which only the set A, and not every factor B, has a representation as a product of indecomposable factors. On the other hand, we can strengthen (iii) by stating that every factor B has, apart from order, just one such representation; and the condition thus obtained alone suffices to characterize the algebras discussed, since it can easily be shown to imply B.1(i). Finally, in C.10(v) we can replace '$b_{m+i} = 0$' by '$a_{m+i} = a_m$'.

Theorem C.11. *If* $\mathfrak{A}$ *is an algebra which satisfies* B.1(i) *and* C.10(i), *and if* $\tau(\mathfrak{A}) = \alpha$, *then*

(i) $\langle \Phi(\alpha), \times, \prod_w \rangle$*—as well as* $\langle \Phi(\alpha), \times \rangle$*—is a G.C.A.;*

(ii) *every type* $\beta \varepsilon \Phi(\alpha)$ *can be represented in the form*

$$\beta = \prod_{i \varepsilon I}{}_w \beta_i$$

where all the types β_i *with* $i \varepsilon I$ *are indecomposable, and this representation is unique, apart from order.*

Proof: (i) can be obtained in an analogous way as the conclusion of C.5; instead of B.7, we apply A.18(ii). Conclusion (ii) can easily be derived from A.4, A.17, and C.10(iii), with the help of 4.38, A.1, A.2, and A.22(iii); to show that the representation is unique, we use A.2 and B.6.

Theorem C.12. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be an algebra which satisfies the following conditions* (i) *and* (ii), *or* (i) *and* (iii)*:*

(i) *if* $a, b \varepsilon A$ *and* $a + b = 0$, *then* $a = b = 0$;

(ii) *every element* $a \varepsilon A$ *can be represented in the form*

$$a = \sum_{i<n} a_i$$

where $n < \infty$ *and all the elements* a_i *with* $i < n$ *are indecomposable in the disjunctive algebra* $\dot{\mathfrak{A}}$ (*i.e.*, $a_i \neq 0$, *and for all* $x, y \varepsilon A$ *the formulas* $a_i = x + y$ *and* $x \cap y = 0$ *imply that* $x = 0$ *or* $y = 0$);

(iii) *if* $a_0, a_1, \cdots, a_i, \cdots, b_0, b_1, \cdots, b_i, \cdots \varepsilon A$, *and if* $a_n = b_n + a_{n+1}$ *and* $b_n \cap a_{n+1} = 0$ *for every* $n < \infty$, *then there is an integer* $m < \infty$ *such that* $a_m = a_{m+i}$ *for* $i = 0, 1, 2, \cdots$.

Under these assumptions $\mathfrak{A}$ *satisfies all the conditions of* B.1 *and* C.10.

PROOF: Assume, e.g., that $\mathfrak{A}$ satisfies C.12(i),(ii). Then, by B.9, $\mathfrak{A}$ satisfies the conditions of B.1. Furthermore, we notice that every element $a \varepsilon A$ which is indecomposable in $\dot{\mathfrak{A}}$ is simple in the sense of C.9. (We apply here A.11(i) and the first formula of A.21(iv). Though this formula was obtained under the assumption that $\mathfrak{A}$ is partially ordered, it is easily seen from 3.1 and A.21(ii) that the formula holds in every algebra $\mathfrak{A}$ to which condition (i) of our theorem applies.) Hence we conclude that $\mathfrak{A}$ satisfies C.10(iv), and therefore also the remaining conditions of C.10. The proof under the assumption that $\mathfrak{A}$ satisfies C.12(i),(iii) is analogous; we use C.10(v) instead of C.10(iv).

Conditions (ii) and (iii) of C.12 are closely related to each other, and in various special classes of algebras they are actually equivalent.

It may be noticed that—in deriving C.12 and theorems upon which it rests—we apply conditions C.12(i)–(iii) only in cases where some of the elements involved belong to disjoint factors. Hence C.12 remains valid if we provide conditions (i)–(iii) with certain further premises of an arithmetical character. For instance, we could include the formula

$$a + b = b + a, \quad \text{or} \quad a_{n+1} + b_n = b_n + a_{n+1},$$

in the hypothesis of (i), or (iii), respectively; this would somewhat improve Theorem C.12 by increasing its applicability to non-commutative algebras.

On the other hand, Theorem C.12 remains valid if (ii) is replaced by a stronger condition (ii′) in which the elements a_i are required to be indecomposable in the algebra $\mathfrak{A}$ itself, and not in $\dot{\mathfrak{A}}$; or if (iii)

is replaced by the ordinary finite chain condition formulated, e.g., in the following way:

(iii′) *if* $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$ *and if* $a_{n+1} \leqq a_n$ *for every* $n < \infty$, *then there is an integer* $m < \infty$ *such that* $a_m = a_{m+i}$ *for* $i = 0, 1, 2, \cdots$.

The modifications just mentioned would simplify, but at the same time considerably weaken, Theorem C.12; for instance, condition (ii′) fails in every lattice and, more generally, in every idem-multiple algebra with at least two distinct elements. As is easily seen, (iii′) implies C.12(i); hence, the conclusions of C.12 apply to every algebra $\mathfrak{A}$ which satisfies the ordinary finite chain condition.[7]

THEOREM C.13. *Let* T *be the class of types of all algebras* $\mathfrak{A}$ *which satisfy* C.12(i),(ii), *or of all algebras* $\mathfrak{A}$ *which satisfy* C.12(i),(iii). *Then*

(i) *conclusion* B.10(i) *holds, and also conclusion* B.10(ii) *for weak products;*

(ii) *the algebra* $\langle \mathrm{T}, \times, \prod_w \rangle$*—as well as* $\langle \mathrm{T}, \times \rangle$*—is a C.A.;*

(iii) *every type* $\alpha \varepsilon \mathrm{T}$ *can be represented in the form*

$$\alpha = \prod_{i \varepsilon I}{}_w \, \alpha_i$$

where all the types α_i *with* $i \varepsilon I$ *are indecomposable, and this representation is unique, apart from order;*

(iv) $\mathfrak{K}$ *being the algebra of cardinal numbers of* 17.3, *and* B *the class of all indecomposable types* $\beta \varepsilon \mathrm{T}$, *we have*

$$\langle \mathrm{T}, \times, \textstyle\prod_w \rangle \cong \mathfrak{K}_s^{\mathrm{B}}.$$

PROOF: (i) can be verified directly. (ii) and (iii) follow from (i), C.11, and C.12. (iv) can be derived from (i) and (iii) by reasoning exactly as in the proof of 18.8; we use weak cardinal powers of types (i.e. weak products of equal types) instead of multiples of relation numbers.

Conclusion C.13(ii) is a rather simple consequence of C.13(iii) or C.13(iv), and similarly C.11(i) follows from C.11(ii). We can repeat here all the remarks made in connection with 18.6–18.8.

[7] Theorem C.12 implies as a particular case a result in Birkhoff [4], p. 616, by which every lattice which satisfies the ordinary finite chain condition has a unique representation described in C.10(iii).

Classes of algebras involved in C.12 and C.13 include various classes of at most denumerable algebras, e.g., those of all at most denumerable G.C.A.'s, of all at most denumerable semigroups which satisfy C.3(i), and of all at most denumerable lattices which are countably complete in the wider sense and in which Theorem 3.12 or the distributive law 3.32 holds. If T is the class of types correlated with any one of these classes of at most denumerable algebras, then, as is easily seen, $\langle \mathrm{T}, \times \rangle$ is one of the algebras (C.A.'s) discussed in 14.10.

In applying C.13(iii) to types of at most denumerable algebras we can, of course, require that the set I be at most denumerable. We shall now give two rather elementary theorems regarding algebras for which this set I proves to be finite.

THEOREM C.14. *Let* $\mathfrak{A} = \langle A, + \rangle$ *be an algebra which satisfies* C.12(i) *and in which there is no infinite sequence of elements* $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$ *with* $a_i \neq 0$ *and* $a_i \cap a_j = 0$ *for* $i < j < \infty$. *Then* $\mathfrak{F}(\mathfrak{A})$ *is a finite disjunctive Boolean algebra.*

PROOF: $\mathfrak{F}(\mathfrak{A})$ is a disjunctive Boolean algebra by B.9. If it were infinite, we could construct, by means of a familiar procedure and in view of A.22(iii), an infinite sequence of factors $A_0, A_1, \cdots, A_i, \cdots$ such that

$$A_i \neq \{0\} \quad \text{and} \quad A_i \cap_f A_j = \{0\} \quad \text{for} \quad i < j < \infty.$$

Hence all the products $A_i \times A_j$ would exist and be factors of $\mathfrak{A}$; and with the help of A.21 we could obtain an infinite sequence of elements $a_0, a_1, \cdots, a_i, \cdots \varepsilon A$ with

$$a_i \neq 0 \quad \text{and} \quad a_i \cap a_j = 0 \quad \text{for} \quad i < j < \infty.$$

(Regarding the applicability of A.21, cf. the proof of C.12.) This conclusion, however, contradicts the hypothesis, and therefore the algebra $\mathfrak{F}(\mathfrak{A})$ must be finite.

THEOREM C.15. *Let* T *be the class of types of all algebras* $\mathfrak{A}$ *which satisfy the hypothesis of* C.14. *Then*

(i) *conclusion* B.10(i) *holds, and also conclusion* B.10(ii) *for finite products;*

(ii) *the algebra* $\langle \mathrm{T}, \times \rangle$ *is a finitely closed G.C.A. and a semigroup;*

(iii) *every type* $\alpha \varepsilon \mathrm{T}$ *can be represented in the form*

$$\alpha = \prod_{i<n} \alpha_i \text{ with } n < \infty$$

where all the types α_i with $i < n$ are indecomposable, and this representation is unique, apart from order;

(iv) *$\mathfrak{J}$ being the algebra of integers of* 14.1, *and* B *the class of all indecomposable types $\beta \varepsilon$ T, we have*

$$\langle \mathrm{T}, \times \rangle \cong \mathfrak{J}_w^{\mathrm{B}}.$$

PROOF: (i) is obvious. To obtain (iii) now, consider an algebra $\mathfrak{A} = \langle A, + \rangle$ satisfying the hypothesis of C.14, and let $\alpha = \tau(\mathfrak{A})$. By C.14, $\mathfrak{F}(\mathfrak{A})$ is a finite disjunctive Boolean algebra. Consequently, as is well known, A can be uniquely represented as a finite product of indecomposable factors. Hence we easily derive the existence and the unicity of the representation of α described in (iii); we apply 4.38, A.1, A.2, A.4, and A.22(iii). Furthermore, by B.10(iii), the algebra $\langle \mathrm{T}, \times \rangle$ is a groupoid. This together with (iii) implies that $\langle \mathrm{T}, \times \rangle$ belongs to the algebras discussed in 14.11. Therefore (ii) and (iv) hold; and the proof is complete.

The conclusions of C.14 and C.15 apply, in particular, to finite algebras $\mathfrak{A}$ with a strong zero element and to the correlated isomorphism types. It has been shown that the conclusions of C.15 apply to isomorphism types of arbitrary finite algebras with a zero element (although the conclusion of C.14 may fail for these algebras); the proof, however, is much more involved than that of C.15.[8]

All the results stated so far in this appendix—except those concerning special G.C.A.'s and semigroups—can be extended to arbitrary algebraic systems $\mathfrak{A}$ formed by a set A, a binary operation $+$, and a (finite or transfinite) sequence of other operations O_0, $O_1, \cdots, O_\xi, \cdots$. All kinds of operations are admitted here: unary, binary, etc., as well as operations on infinite, or even transfinite, sequences. The set A is assumed to contain an element 0 which is both the zero element for the operation $+$ and an IDEM-POTENT ELEMENT for all the operations O_ξ; i.e., if

$$a_0 = a_1 = a_2 = \cdots = 0,$$

then

$$O_\xi(a_0, a_1, a_2, \cdots) = 0.$$

[8] The result is well known in its application to finite groups; cf., for instance, Speiser [1], pp. 135 f., where bibliographic references are given. In its general form the result is established in Jónsson-Tarski [1].

The formulations of definitions and theorems, as well as the proofs of the latter, remain practically unchanged. Theorem A.11 must, however, be provided with an additional condition (iv) which can be formulated as follows:

(iv) *for every operation* O_ξ, *if* $b_i, b_i^{(0)}, b_i^{(1)}, \cdots \varepsilon B_i$ *for* $i = 0, 1, \cdots, n - 1$, *then the formula*

$$\sum_{i<n} b_i = O_\xi(\sum_{i<n} b_i^{(0)}, \sum_{i<n} b_i^{(1)}, \cdots)$$

implies that $b_i = O_\xi(b_i^{(0)}, b_i^{(1)}, \cdots)$ *for* $i = 0, 1, \cdots, n - 1$), *and conversely.*

D. CARDINAL PRODUCTS OF RELATION NUMBERS

In this last section we want to show how the results obtained for isomorphism types of algebras can be applied to another kind of isomorphism types, in fact, to the relation numbers discussed in §18.

Given any relations R_i correlated with elements i of an arbitrary set I, we understand by the STRONG CARDINAL PRODUCT OF R_i the set R of all ordered couples $\langle f, g\rangle$ where f and g are functions subjected to the following conditions:

$$D(f) = D(g) = I, \quad \text{and} \quad \langle f(i), g(i)\rangle \;\varepsilon\; R_i \quad \text{for every} \quad i \;\varepsilon\; I.$$

We put in symbols

$$R = \prod_{i \varepsilon I}{}_s R_i\,;$$

and in particular,

$$R = S \times T,$$

in case I consists of two numbers 0 and 1, $R_0 = S$, and $R_1 = T$.[9]

Two products $\prod_{i\varepsilon I}{}_s R_i$ and $\prod_{i\varepsilon I}{}_s S_i$ are clearly similar if the corresponding relations R_i and S_i are similar. Hence we can define the operation of STRONG CARDINAL MULTIPLICATION OF RELATION NUMBERS in such a way that

$$\prod_{i\varepsilon I}{}_s \rho(R_i) = \rho\Big(\prod_{i\varepsilon I}{}_s R_i\Big)$$

and in particular

$$\rho(S) \times \rho(T) = \rho(S \times T).$$

By the ZERO ELEMENT OF A RELATION R we understand an element z which is the only element in $F(R)$ satisfying the condition: $z\,R\,x$ for every $x \;\varepsilon\; F(R)$; we denote this element z by '0'. If we restrict ourselves to relations with zero elements, we can define for them and for their relation numbers the notions of a (plain) CARDINAL PRODUCT and of a WEAK CARDINAL PRODUCT by modifying in an

[9] The notion of the cardinal product of two partially ordering relations was introduced in Birkhoff [1], p. 287.

obvious way the definition of a strong cardinal product; we use the symbols

$$\prod_{i \varepsilon I} R_i \quad \text{and} \quad \prod_{i \varepsilon I}{}_w R_i .$$

The common relation number of all relations R consisting of one couple $\langle x, x \rangle$ (with two identical terms) is called the UNIT NUMBER, 1; this relation number has the properties formulated in 12.7. A relation number ρ is called (MULTIPLICATIVELY) INDECOMPOSABLE if $\rho \neq 1$ and if, for arbitrary relation numbers σ and τ, $\rho = \sigma \times \tau$ implies that $\sigma = 1$ or $\tau = 1$. The definition of the FACTOR RELATION, A.1, can be carried over automatically to relation numbers.

As is well known, a relation R is called REFLEXIVE if $x \, R \, x$ for every $x \, \varepsilon \, F(R)$; it is called ANTISYMMETRIC if $x \, R \, y$ and $y \, R \, x$ always imply $x = y$. If we confine ourselves to relations R which have a zero element and are reflexive and antisymmetric, then the discussion of cardinal products of correlated relation numbers reduces entirely to that of cardinal products of isomorphism types correlated with a certain class of algebras. This reduction is based upon the following:

DEFINITION D.1. *R being an arbitrary relation, an element $a \, \varepsilon \, F(R)$ is called the R-SUM OF THE ELEMENTS b, $c \, \varepsilon \, F(R)$ (or the LEAST UPPER BOUND OF b AND c UNDER R), in symbols,*

$$a = b +_R c,$$

if a is the only element such that $b \, R \, a$, $c \, R \, a$, and such that, for every x, the formulas $b \, R \, x$ and $c \, R \, x$ imply $a \, R \, x$. The algebra

$$\mathfrak{A}_R = \langle F(R), +_R \rangle$$

is referred to as the ALGEBRA GENERATED BY R.

THEOREM D.2. *Let P be the class of all relation numbers correlated with relations R which have a zero element and are reflexive and antisymmetric; and let P' be the class of all types correlated with algebras $\mathfrak{A}_R$ which are generated by such relations R. For any given relation number σ, let $\varphi(\sigma)$ be the common type of all algebras $\mathfrak{A}_R$ such that $\rho(R) = \sigma$. Then*

(i) *φ maps P onto P' in a one-to-one way;*

(ii) *if $\sigma_i \, \varepsilon \, \mathrm{P}$ for every $i \, \varepsilon \, I$, then*

$$\varphi\left(\prod_{i \varepsilon I} \sigma_i\right) = \prod_{i \varepsilon I} \varphi(\sigma_i);$$

(iii) *if $\sigma \varepsilon$ P and if α_i with $i \varepsilon I$ are types of algebras for which*

$$\sigma = \prod_{i \varepsilon I} \alpha_i ,$$

then there are relation numbers $\sigma_i \varepsilon$ P such that

$$\sigma = \prod_{i \varepsilon I} \sigma_i , \quad \text{and} \quad \varphi(\sigma_i) = \alpha_i \quad \text{for every} \quad i \varepsilon I.$$

Similarly for strong and weak products.

The proof is elementary and almost mechanical. We make use of the obvious facts that the zero element of a relation R with $\rho(R) \varepsilon$ T is at the same time the zero element of the correlated algebra $\mathfrak{A}_R$; and that the relation R can be characterized in terms of the operation $+_R$ (in fact, R coincides both with the $\leqq$ relation and the absorption relation of the algebra $\mathfrak{A}_R$). The restriction to relations having a zero element is superfluous in the case of strong products.

Theorem D.2 has several consequences.

Corollary D.3. *Under the assumptions of* D.2, *the algebras* $\langle \mathrm{P}, \times \rangle$, $\langle \mathrm{P}, \times, \prod \rangle$, *and* $\langle \mathrm{P}, \times, \prod_w \rangle$ *are isomorphic with the algebras* $\langle \mathrm{P}', \times \rangle$, $\langle \mathrm{P}', \times, \prod \rangle$, *and* $\langle \mathrm{P}', \times, \prod_w \rangle$, *respectively.*

Proof: by 6.1 and D.2.

Theorem D.4. *Let* P *be the class of all relation numbers correlated with relations R which have a zero element and are reflexive and antisymmetric. Then all conclusions of* B.10 *apply to* T = P.[10]

Proof: Conclusions B.10(i),(ii) for T = P can be verified directly; hence, by D.2, we conclude that they also apply to the class T′. Furthermore, we notice that for every relation R with $\rho(R)$ ε P, the correlated algebra $\mathfrak{A}_R$ satisfies the hypothesis of B.8; thus, the class P′ of D.2 is a subclass of the class T of B.10. Hence, by applying B.10 and D.2 (or D.3), we arrive at the remaining conclusions.

Among relations R discussed in D.2 and D.4, those which generate lattices deserve special attention. They can be characterized as partially ordering (i.e., reflexive, antisymmetric, and transitive) relations, with a zero element, and in which any two elements b and c in $F(R)$ have both a least upper bound under R in the sense of D.1

[10] For a more special class of relations—in fact, for partially ordering relations with a zero element and a unit element—the essential part of this result was established in Birkhoff [3], p. 23.

and an analogously defined GREATEST LOWER BOUND UNDER R. We now have

THEOREM D.5. *Let* P *be the class of relation numbers of all relations generating countably distributive lattices, or Boolean algebras, which are countably complete in the wider sense. Then the conclusions of* C.4 *apply to* T = P.

THEOREM D.6. *Let* P *be the class of relation numbers of all relations generating countably complete and countably distributive lattices, or countably complete Boolean algebras. Then the conclusions of* C.8 *apply to* T = P.

The proofs of D.5 and D.6 are analogous to that of D.2; we apply 15.27, 15.28, and D.2.

THEOREM D.7. *Let* P *be the class of all relation numbers correlated with those relations* R *which have a zero element, are reflexive, and satisfy the following condition;*

(i) *if* $a_{n+1} \, R \, a_n$ *for every* $n < \infty$, *then there is an integer* $m < \infty$ *such that* $a_m = a_{m+i}$ *for* $i = 0, 1, 2, \cdots$.

Then the conclusions of C.13 *apply to* T = P.

PROOF: analogous to D.4 with B.10 replaced by C.13. We notice that the finite chain condition D.7(i) implies the antisymmetry of R, and that therefore D.2 can be applied.

THEOREM D.8. *Let* P *be the class of all relation numbers correlated with those relations* R *which have a zero element, are reflexive and antisymmetric, and satisfy the following condition:*

(i) *there is no infinite sequence of elements* $a_0, a_1, \cdots, a_i, \cdots$ *such that* $a_i \neq 0$ *for every* $i < \infty$, *and such that* $x \, R \, a_i$ *and* $x \, R \, a_j$ *imply* $x = 0$ *for every element* x *and for* $i < j < \infty$.

Then the conclusions of C.15 *apply to* T = P.

PROOF: analogous to D.4, with B.10 replaced by C.15.

It would be interesting to extend the preceding theorems D.4–D.8 to wider classes of relation numbers. In the case of D.4 and D.8 certain extensions are known which are not, however, far-reaching. They have been obtained by weakening the condition of antisymmetry. In fact, we have:

THEOREM D.9. *Let* P *be the class of all relation numbers correlated with relations* R *which are reflexive and satisfy the following condition:*

(i) *there is an element* $z \varepsilon F(R)$ *such that* $z\,R\,x$ *for every* $x \varepsilon F(R)$, *and such that* $x\,R\,z$ *always implies* $x = z$.
Then all conclusions of B.10 *apply to* $T = P$.

PROOF: The problem reduces to showing that B.10(iv) holds for P, since the remaining conclusions either are obvious or easily follow from B.10(iv). We cannot apply D.2 here, but we can use a method of reasoning which is analogous to that applied in the proofs of B.8–B.10. Roughly speaking, we proceed in the following way. Given any relation numbers ρ_i, $\sigma_j \varepsilon$ P such that

$$\prod_{i \varepsilon I} \rho_i = \prod_{j \varepsilon J} \sigma_j ,$$

we consider a relation R with

$$\rho(R) = \prod_{i \varepsilon I} \rho_i = \prod_{j \varepsilon J} \sigma_j .$$

Using now the idea of A.3, we represent R as a kind of a direct product of subrelations R_ι with $\rho(R_\iota) = \rho_\iota$, and also of subrelations S_j with $\rho(S_j) = \sigma_j$. We put

$$\tau_{i,j} = \rho(R_i \cap S_j),$$

and we show that

$$\rho_i = \prod_{j \varepsilon J} \tau_{\iota,j} \quad \text{and} \quad \sigma_j = \prod_{i \varepsilon I} \tau_{i,j} \quad \text{for } i \varepsilon I \text{ and } j \varepsilon J.$$

THEOREM D.10. *Let* P *be the class of all relation numbers correlated with relations* R *which are reflexive and satisfy* D.8(i) *and* D.9(i). *Then the conclusions of* C.15 *apply to* $T = P$.

PROOF: Every relation number $\rho \varepsilon$ P can be represented as a product of finitely many multiplicatively indecomposable numbers. For otherwise we could construct two infinite sequences of relation numbers ρ_i and σ_ι such that

$$\rho_0 = \rho, \quad \sigma_n \neq 1 \quad \text{and} \quad \rho_n = \sigma_n \times \rho_{n+1} \quad \text{for} \quad n = 0, 1, 2, \cdots ;$$

and this, as is easily seen, contradicts D.8(i). On the other hand, ρ cannot have two different representations of this kind in view of D.9. Thus, C.15(iii) applies to P. Other conclusions either are obvious or easily follow from C.15(iii); cf. the proof of C.15.

By D.10, the conclusion of C.15 apply, in particular, to all relation numbers correlated with finite relations R which are reflexive and satisfy D.9(i). This result cannot be extended, however, to relation

numbers of arbitrary finite relations. In fact, let R be the set of two couples $\langle 0, 0\rangle$ and $\langle 1, 1\rangle$, and let S be the set of two couples $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$. We then have

$$\rho(R) \neq \rho(S) \quad \text{and} \quad \rho(R) \times \rho(S) = \rho(S) \times \rho(S);$$

hence the relation number

$$\alpha = \rho(R) \times \rho(S)$$

does not satisfy C.15(iii).[11] Instead of relations R and S, we can consider here algebras $\mathfrak{A}' = \langle A, +'\rangle$ and $\mathfrak{A}'' = \langle A, +''\rangle$ where A is the set of two numbers, 0 and 1, and where

$$0 +' n = 0, \quad 1 +' n = 1, \quad 0 +'' n = 1, \quad \text{and} \quad 1 +'' n = 0$$
$$\text{for} \quad n = 0, 1.$$

These are, however algebras without zeros, in which we are not interested here.

In conclusion we want to notice that all the more profound results of the arithmetic of isomorphism types and relation numbers which are implicitly contained in various theorems of this appendix have a rather narrow range of application. Consider, for instance, the following two properties of cardinal products:

(i) *If* $\alpha \times \beta \times \gamma = \gamma$, *then* $\alpha \times \gamma = \beta \times \gamma = \gamma$.
(ii) *If* $\alpha \times \alpha = \beta \times \beta$, *then* $\alpha = \beta$.

(i) implies that the factor relation is an antisymmetric, and hence a partially ordering, relation; disregarding the multiplicative notation, we recognize in (i) a particular case of the absorption theorem 1.47 from Part I. Similarly, (ii) is a particular case of the cancellation law 2.34. Both (i) and (ii) clearly hold for isomorphism types and relation numbers which have a unique representation as finite or infinite products of indecomposable factors (cf. C.13, C.15, C.6, D.8, and D.10). Moreover, they apply to types and numbers belonging to any class T which constitutes a C.A. or a G.C.A. under cardinal multiplication $\times$ (cf. C.8 and D.5); and (i) holds, in addition, in case the class T is only an R.A. under the same operation (cf. C.4 and D.4). Some other cases are also known in which (i) holds; for instance, using certain properties of cardinal addition $+$ discussed

[11] A similar example was found independently by J. C. C. McKinsey.

in §18, and certain elementary theorems which connect $+$ with $\times$, we can easily show that (i) applies whenever γ is a relation number of the form $\gamma = 1 + \delta$.[12]

At any rate, the isomorphism types and relation numbers for which (i) and (ii) have been proved to hold are of a rather special nature. Hence the problem arises of constructing examples which would show that (i) or (ii) do not apply to arbitrary types and numbers, or that they may fail in various familiar classes of such types and numbers. This problem turns out to be more involved than might be expected. In fact, no such examples seem to be known in the domain of algebras and relations with zero elements. On the other hand, an example of two relations R and S without zeros whose relation numbers do not satisfy (ii) can easily be obtained: we take for R the set consisting of all couples $\langle n, n + 1 \rangle$ where n is a finite integer, and for S the relation $R \times R$. A simple transformation of this example leads to two algebras $\mathfrak{A}$ and $\mathfrak{B}$ without zeros whose types do not satisfy (ii). In fact, we put

$$\mathfrak{A} = \langle A, \oplus \rangle \quad \text{and} \quad \mathfrak{B} = \mathfrak{A} \times \mathfrak{A}$$

where A is the set of all finite integers, and $\oplus$ the operation defined by the formula

$$n \oplus p = n + 1 \quad \text{for} \quad n < \infty \quad \text{and} \quad p < \infty.$$

The problem of finding examples of this kind among certain familiar classes of algebraic systems, such as groups or Boolean algebras, seems to be especially interesting. The fact that this problem is open shows that our knowledge of the arithmetic of isomorphism types is still rather limited.

[12] This result as well as the examples given below were found and communicated to the authors by J. C. C. McKinsey.

BIBLIOGRAPHY

This bibliography contains only the titles of works actually referred to in the text.

R. Baer

[1] *The homomorphism theorems for loops.* ***American journal of mathematics,*** vol. 47, 1945, pp. 450–60.

S. Banach

[1] *Sur le problème de la mesure.* ***Fundamenta mathematicae,*** vol. 4, 1923, pp. 7–33.

[2] *Un théorème sur les transformations biunivoques.* ***Fundamenta mathematicae,*** vol. 6, 1924, pp. 236–9.

S. Banach and A. Tarski

[1] *Sur la décomposition des ensembles de points en parties respectivement congruentes.* ***Fundamenta mathematicae,*** vol. 6, 1924, pp. 244–77.

G. Bergmann

[1] *Zur Axiomatik der Elementargeometrie.* ***Monatshefte für Mathematik und Physik,*** vol. 36, 1929, pp. 269–84.

P. Bernays

[1] *A system of axiomatic set theory.* Part I, ***Journal of symbolic logic,*** vol. 2, 1937, pp. 65–77. Part II, ibid., vol. 6, 1941, pp. 1–17. Part III, ibid., vol. 7, 1942, pp. 65–89. Part IV, ibid., vol. 7, 1942, pp. 133–45. Part V, ibid., vol. 8, 1943, pp. 89–106.

F. Bernstein

[1] *Untersuchungen aus der Mengenlehre.* ***Mathematische Annalen,*** vol. 61, 1905–6, pp. 117–55.

G. Birkhoff

[1] *Generalized arithmetic.* ***Duke mathematical journal,*** vol. 9, 1942, pp. 283–302.

[2] *Lattice-ordered groups.* ***Annals of mathematics,*** vol. 43, 1942, pp. 298–331.

[3] *Lattice theory.* New York, 1940, IV + 155 pp. [***American Mathematical Society Colloquium publications,*** vol. 25.]

[4] *On the lattice theory of ideals.* ***Bulletin of the American Mathematical Society,*** vol. 40, 1935, pp. 613–19.

G. Birkhoff and S. MacLane.

[1] *A survey of modern algebra.* New York, 1941, VII + 450 pp.

G. Boole

[1] *An investigation of the laws of thought, on which are founded the mathematical theories of logic and probabilities.* London, 1854, V + IV + 424 pp. [Reprinted as vol. 2 of ***George Boole's collected logical works,*** Chicago and London, 1916.]

H. FITTING

[1] *Über die Existenz gemeinsamer Verfeinerungen bei directen Productzerlegungen einer Gruppe.* ***Mathematische Zeitschrift,*** vol. 41, 1936, pp. 380–95.

O. N. GOLOWIN

[1] *Mnozhiteli bez tsentrov v priamykh razlozheniyakh grupp.* (*On factors without centres in direct decompositions of groups.*) ***Matematicheskiy sbornik*** (***Recueil mathématique***), vol. 48, 1939, pp. 423–6.

F. HAUSDORFF

[1] *Grundzüge der Mengenlehre.* Leipzig, 1914. VIII + 476 pp.

E. V. HUNTINGTON

[1] *A complete set of postulates for the theory of absolute continuous magnitude.* ***Transactions of the American Mathematical Society,*** vol. 3, 1902, pp. 264–79.

B. JÓNSSON AND A. TARSKI

[1] *Direct decompositions of finite algebraic systems.* Notre Dame, Indiana, 1947, VI + 64 pp. [***Notre Dame mathematical lectures,*** vol. 5.]

B. KNASTER

[1] *Un théorème sur les fonctions d'ensembles.* ***Annales de la Société Polonaise de Mathématique,*** vol. 6, 1927, pp. 133–4.

D. KÖNIG

[1] *Sur les correspondances multivoques des ensembles.* ***Fundamenta mathematicae,*** vol. 8, 1926, pp. 114–34.

A. KORSELT

[1] *Über einen Beweis des Äquivalenzsatzes.* ***Mathematische Annalen,*** vol. 70, 1910, pp. 294–6.

C. KURATOWSKI

[1] *Sur une généralisation de la notion d'homéomorphie.* ***Fundamenta mathematicae,*** vol. 22, 1934, pp. 206–20.

[2] *Topologie I.* Warsaw, 1933, IX + 285 pp. [***Monografje matematyczne,*** vol. 3.]

[3] *Une propriété des correspondances biunivoques.* ***Fundamenta mathematicae,*** vol. 6, 1924, pp. 240–43.

A. LINDENBAUM AND A. TARSKI

[1] *Communication sur les recherches de la théorie des ensembles.* ***Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie, Classe III,*** vol. 19, 1926, pp. 299–330.

J. VON NEUMANN

[1] *Lectures on continuous geometry, 1936–1937.* Ann Arbor, 1937 [planographed], IV + 166 + 36 pp.

[2] *Zur allgemeinen Theorie des Masses.* ***Fundamenta mathematicae,*** vol. 13, 1929, pp. 73–116.

F. RIESZ

[1] *Sur quelques notions fondamentales dans la théorie générale des opérations linéaires.* ***Annals of mathematics,*** vol. 41, 1940, pp. 174–206.

A. Schoenflies

[1] *Entwickelung der Mengenlehre und ihrer Anwendungen.* Part I. Leipzig and Berlin, 1913, X + 288 pp.

W. Sierpiński

[1] *Leçons sur les nombres transfinis.* Paris, 1928, VI + 240 pp. [***Collection de monographies sur la théorie des fonctions.***]

[2] *Sur l'égalité* 2m = 2n *pour les nombres cardinaux.* ***Fundamenta mathematicae,*** vol. 3, 1922, pp. 1–6.

A. Speiser

[1] *Die Theorie der Gruppen von endlicher Ordnung.* 2nd edition. Berlin, 1927, IX + 251 pp. [***Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen,*** vol. 5.]

M. H. Stone

[1] *Postulates for Boolean algebras and generalized Boolean algebras.* ***American journal of mathematics,*** vol. 57, 1935, pp. 703–32.

[2] *The theory of representations for Boolean algebras.* ***Transactions of the American Mathematical Society,*** vol. 40, 1936, pp. 37–111.

[3] *Topological representations of distributive lattices and Brouwerian logics.* ***Časopis pro pěstování matematiky a fysiky,*** vol. 67, 1937–8, pp. 1–25.

A. Tarski

[1] *Algebraische Fassung des Massproblems.* ***Fundamenta mathematicae,*** vol. 31, 1938, pp. 47–66.

[2] *Geschichtliche Entwickelung und gegenwärtiger Zustand der Gleichmächtigkeitstheorie und der Kardinalzahlarithmetik.* ***Księga pamiątkowa Pierwszego Polskiego Zjazdu Matematycznego Lwów 1927,*** Kraków, 1929, pp. 48–54.

[3] *Grundzüge des Systemenkalküls.* Part I. ***Fundamenta mathematicae,*** vol. 25, 1935, pp. 503–26.

[4] *Les fonctions additives dans les classes abstraites et leur applications au problème de la mesure.* ***Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie, Classe III,*** vol. 22, 1929, pp. 114–17.

[5] *Quelques théorèmes généraux sur les images d'ensembles.* ***Annales de la Société Polonaise de Mathématique,*** vol. 6, 1928, pp. 132–3.

[6] *Sur les groupes d'Abel ordonnés.* ***Annales de la Société Polonaise de Mathématique,*** vol. 7, 1929, pp. 267–8.

[7] *Über additive and multiplicative Mengenkörper und Mengenfunktionen.* ***Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie, Classe III,*** vol. 30, 1937, pp. 151–81.

[8] *Über Äquivalenz der Mengen in bezug auf eine beliebige Klasse von Abbildungen.* ***Atti del Congresso Internazionale dei Matematici Bologna 1928,*** Bologna, 1930, vol. 2, pp. 186–205.

[9] *Über das absolute Mass linearer Punktmengen.* ***Fundamenta mathematicae,*** vol. 30, 1938, pp. 218–34.

[10] *Zur Grundlegung der Boole'schen Algebra. I.* ***Fundamenta mathematicae,*** vol. 24, 1935, pp. 177–98.

B. L. van der Waerden

[1] *Moderne Algebra*. 2nd edition. Vol. 1, Berlin, 1937, IX + 272 pp. [***Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen,*** vol. 33.]

A. N. Whitehead and B. Russell

[1] *Principia mathematica*. 2nd edition. Vol. 1, Cambridge, 1925, XLVI + 674 pp. Vol. 2, Cambridge, 1927, XXXI + 742 pp.

E. Zermelo

[1] *Über die Addition transfiniter Cardinalzahlen*. ***Nachrichten von der Königlichen Göttingen Gesellschaft der Wissenschaften zu Göttingen,*** 1901, pp. 34–8.

INDEX

The following remarks may prove helpful in the use of this index:

1. Dashes replace words. Consider, e.g., the paragraph below beginning with the term 'Addition.' It is divided into segments by semicolons. The initial dash in the second segment replaces the first word of the preceding segment, i.e., the word 'Addition'; the later dash in the same segment replaces the first word of this segment, i.e., again the word 'Addition'; the initial two dashes in the third segment replace the first two words in its preceding segment, i.e., 'Addition of'; and so on.

2. In brackets following a term we give synonyms, symbols, and abbreviations of this term actually used in the book. In parentheses we put the part of a term which is sometimes omitted in the text.

3. The word 'algebra' is used as a generic term comprehending cardinal algebras, semigroups, lattices, etc. Thus, e.g., to find information on countably distributive lattices, look under 'Countably distributive algebra.'

ADDENDA

In the following remarks we should like to point out some minor errors and inexactitudes which were found in the text, after the manuscript had been set up in page proof, and kindly communicated to the author by Mrs. Anne C. Davis (University of California, Berkeley). We also take this opportunity to mention some recent results and publications which are closely related to topics discussed in this book.

To page 3. According to our conception of an algebraic system, the fundamental operations of an algebra may be of an entirely arbitrary nature. In particular, the binary operation $+$ in an algebra $\langle A, +, \sum \rangle$ may be defined for some couples of elements which are not in A, and when performed on a couple for which it is defined it may yield an element outside of A; the same applies *mutatis mutandis* to the infinite operation $\sum$. This possibility is not excluded by definitions of special kinds of algebras, e.g., C. A.'s, G. C. A.'s, or Boolean algebras.

In consequence, certain theorems given in the book prove to be formulated inexactly and, when literally taken, are simply wrong. Consider, e.g., Corollary 8.9 on p. 104. In the idem-multiple G. C. A. $\mathfrak{A} = \langle A, +, \sum \rangle$ the operation $+$ may be defined for some couples of elements outside of A; on the other hand, by Definition 3.2 on p. 40, the operation $\cup$ in the correlated algebra $\breve{\mathfrak{A}} = \langle A, \cup, \bigcup \rangle$ is defined only for couples in A; hence, the algebras $\mathfrak{A}$ and $\breve{\mathfrak{A}}$ may not be logically identical. Analogous remarks apply, e.g., to Theorems 8.19 and 8.20 on pp. 107 f., or 15.17 and 15.18 on p. 209.

To eliminate these inexactitudes, we can agree to consider as algebras only those systems $\langle A, +, \sum \rangle$ in which the operations $+$ and $\sum$ are defined exclusively for couples or sequences of elements in A and yield always new elements in A. This agreement, however, necessitates some slight modifications in other formulations of the text, in particular, in those involving the notion of a subalgebra; for, as a consequence of the agreement, the operations $+$ and $\sum$ in an algebra $\langle A, +, \sum \rangle$ and in its subalgebra $\langle B, +, \sum \rangle$ can no longer be regarded as identical. The following supplementary agree-

ment helps us avoid unnecessary complications in this connection. We continue to use the notion of an ALGEBRA CONSTITUTED BY A SET A AND BY FUNDAMENTAL OPERATIONS $+$ AND $\sum$ without imposing any restrictions on the operations involved. We identify this notion, however, not simply with the ordered triple $\langle A, +, \sum \rangle$, but with the ordered triple $\langle A, +', \sum' \rangle$ where $+'$ is the operation defined as follows:

$a +' b = a + b$ whenever a, b, and $a + b$ are in A, and otherwise

$$a +' b \text{ does not exist;}$$

and where $\sum'$ is the operation analogously defined in terms of $\sum$. In view of this agreement, we can say that an algebra $\mathfrak{A}$ and its subalgebra $\mathfrak{B}$ are constituted by the same fundamental operations though by different sets of elements; at the same time we can state in certain cases that two algebras $\mathfrak{A}$ and $\mathfrak{B}$ (e.g., an idem-multiple G. C. A. $\mathfrak{A}$ and the correlated algebra $\breve{\mathfrak{A}}$) are identical although their fundamental operations do not coincide.

To page 46. The formulation of condition (ii) in Theorem 3.16 is not quite exact. An exact formulation is

(ii) *if* $a \leqq x$ *and if* $x \leqq a_i$ *for every* $i < n$, *then* $x = a$.

Clearly, the two formulations are equivalent unless $n = 0$. An analogous remark applies to Theorem 3.17 on p. 47.

To page 64. It is tacitly assumed in Theorem 4.45 that the elements a_i form an n-termed sequence, and the elements b_j a p-termed sequence.

To page 80. Theorem 6.6 as formulated in the text is probably false; it applies to C. A.'s, but not to G. C. A.'s. It is easily seen, however, that 6.6 holds also for G. C. A.'s under the additional assumption that the relation R is finitely refining; and this is just the assumption under which the theorem in question is applied in the text (e.g., in the proof of 6.10 on p. 81).

The formulation of Definition 6.7, also on p. 80, is not quite clear. The relation R is finitely refining if, for all a, a_1, a_2, $b \in A$, the formulas

$$a = a_1 + a_2 \text{ and } a\,R\,b$$

imply the existence of elements b_1 and b_2 such that

$$b = b_1 + b_2, \text{ and } a_i\,R\,b_i \text{ for } i = 1, 2,$$

while the formulas

$$a = a_1 + a_2 \text{ and } b\,R\,a$$

imply the existence of elements b_1 and b_2 such that

$$b = b_1 + b_2, \text{ and } b_i\,R\,a_i \text{ for } i = 1, 2.$$

To page 100. The remark following Theorem 7.17 is wrong; the theorem does apply to strong (though not to weak) cardinal products. This somewhat weakens the validity of the remarks on p. 84 explaining why 'the strong product appears to be a less appropriate element in dealing with C. A.'s than the plain product.'

To page 114. In the remarks following Theorem 9.18 the algebras $\mathfrak{A}$ and $\mathfrak{B}$ are assumed to be G. C. A.'s.

To page 199. A set of necessary and sufficient conditions for the existence of an infinitely additive and strictly positive measure in countably complete and atomless (disjunctive) Boolean algebras is given in the paper of D. Maharam, *An algebraic characterization of measure algebras*, ***Annals of mathematics***, vol. 48, 1947, pp. 154–67.

To pages 239 ff. The problems discussed on these pages are analyzed in greater detail in the author's paper, *Axiomatic and algebraic aspects of two theorems on sums of cardinals*, to appear in ***Fundamenta mathematicae***, vol. 35, pp. 79–104. In particular, a direct proof of Theorems I–III on p. 241 can be found there.

To page 272. Two new conditions each of which is necessary and sufficient for the factor algebra $\mathfrak{F}(\mathfrak{A})$ of an algebra $\mathfrak{A} = \langle A, + \rangle$ to be a disjunctive Boolean algebra have recently been found by J. M. G. Fell (University of California, Berkeley). We agree to understand by the center of the algebra the set of all elements $z \in A$ satisfying conditions (*i*) and (*ii*) given at the bottom of p. 281 (see Jónsson-Tarski [1], pp. 17 f. and 24). The conditions read as follows:

I. *If* $A = B \times C = B \times D$, *then* $C = D$.

II. *If* $A = B \times C$, *then the subalgebra consisting only of* 0 *is the unique subalgebra of* B *which is a homomorphic image of* C *and which is included in the center of* $\mathfrak{A}$.

The advantage of these conditions lies in the fact that they can easily be applied in practice to comprehensive classes of algebraic systems. For instance, by restricting ourselves to groups, we readily see that condition I is satisfied by all cyclic groups while condition II is satisfied by all centerless groups and by all groups which coincide with their commutator groups. Hence every group belonging to any of these three classes has a Boolean factor algebra.